

# Unbounded normal operators in octonion

## Hilbert spaces and their spectra

Ludkovsky S.V.

15 February 2012

### Abstract

Affiliated and normal operators in octonion Hilbert spaces are studied. Theorems about their properties and of related algebras are demonstrated. Spectra of unbounded normal operators are investigated.<sup>1</sup>

## 1 Introduction

Unbounded normal operators over the complex field have found many-sided applications in functional analysis, differential and partial differential equations and their applications in the sciences [4, 11, 12, 14, 31]. On the other hand, hypercomplex analysis is fast developing, particularly in relation with problems of theoretical and mathematical physics and of partial differential equations [2, 7, 9]. The octonion algebra is the largest division real algebra in which the complex field has non-central embeddings [3, 1, 13]. It is intensively used especially in recent years not only in mathematics, but also in applications [5, 10, 8, 15, 16].

In previous works analysis over quaternion and octonions was developed and spectral theory of bounded normal operators and unbounded self-adjoint

---

<sup>1</sup>key words and phrases: non-commutative functional analysis, hypercomplex numbers, quaternion skew field, octonion algebra, operator, operator algebra, spectra, spectral measure, non-commutative integration

Mathematics Subject Classification 2010: 30G35, 17A05, 17A70, 47A10, 47L30, 47L60

operators was described [18, 19, 20, 21, 22]. Some results on their applications in partial differential equations were obtained [23, 24, 25, 26, 27]. This work continues previous articles and uses their results. The present paper is devoted to unbounded normal operators and affiliated operators in octonion Hilbert spaces, that was not yet studied before.

Frequently in practical problems, for example, related with partial differential operators, spectral theory of unbounded normal operators is necessary. This article contains the spectral theory of unbounded affiliated and normal operators. Notations and definitions of papers [18, 19, 20, 21, 22] are used below. The main results of this paper are obtained for the first time.

## 2 Affiliated and normal operators

**1. Definitions.** Let  $X$  be a Hilbert space over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ , and let  $\mathbf{A}$  be a von Neumann algebra contained in  $L_q(X)$ . We say that an operator  $Q \in L_q(X)$  quasi-commutes with  $\mathbf{A}$  if the algebra  $\text{alg}_{\mathcal{A}_v}(Q, B)$  over  $\mathcal{A}_v$  generated by  $Q$  and  $B$  is quasi-commutative for each  $B \in \mathbf{A}$ . A closed  $\mathbf{R}$  homogeneous  $\mathcal{A}_v$  additive operator  $T$  with a dense  $\mathcal{A}_v$  vector domain  $\mathcal{D}(T) \subset X$  is said to be affiliated with  $\mathbf{A}$ , when

$$(1) \quad U^*TUx = Tx$$

for every  $x \in \mathcal{D}(T)$  and each unitary operator  $U \in L_q(X)$  quasi-commuting with  $\mathbf{A}$ . The fact that  $T$  is affiliated with  $\mathbf{A}$  is denoted by  $T\eta\mathbf{A}$ .

An  $\mathcal{A}_v$  vector subspace  $\bigcup_n {}_nF\mathcal{D}(T)$  is called a core of an operator  $T$ , if  ${}_nF$  is an increasing sequence of  $\mathcal{A}_v$  graded projections and  $\bigcup_n {}_nF\mathcal{D}(T)$  is dense in  $\mathcal{D}(T)$ .

**2. Note.** Definition 1 implies that  $U\mathcal{D}(T) = \mathcal{D}(T)$ . If  $V$  is a dense  $\mathcal{A}_v$  vector subspace in  $\mathcal{D}(T)$  and  $T|_V\eta\mathbf{A}$ , then  $T\eta\mathbf{A}$ . Indeed,  $\lim_n Uy^n = Uy$  for each unitary operator  $U \in L_q(X)$  and every sequence  $y^n \in V$  converging to a vector  $y \in \mathcal{D}(T)$  and with  $\lim_n Ty^n = Ty$ . Therefore, the limit  $\lim_n T Uy^n = \lim_n UT y^n = UT y$  exists. The operator  $T$  is closed, hence  $Uy \in \mathcal{D}(T)$  and  $UT y = T Uy$ . Thus  $\mathcal{D}(T) \subset U^*\mathcal{D}(T)$ . The proof above for  $U^*$  instead of  $U$  gives the inclusion  $\mathcal{D}(T) \subset U\mathcal{D}(T)$ , consequently,  $U\mathcal{D}(T) = \mathcal{D}(T)$  and hence  $\mathcal{D}(U^*TU) = \mathcal{D}(T)$  and  $T Uy = UT y$  for every  $y \in \mathcal{D}(T)$ .

For an  $\mathbf{R}$  homogeneous  $\mathcal{A}_v$  additive (i.e. quasi-linear) operator  $A$  in  $X$  with an  $\mathcal{A}_v$  vector domain  $\mathcal{D}(A)$  the notation can be used:

- (1)  $Ax = \sum_j A^{i_j} x_j$  for each
- (2)  $x = \sum_j x_j i_j \in \mathcal{D}(A) \subset X$  with  $x_j \in X_j$  and
- (3)  $A^{i_j} x_j := A(x_j i_j)$  for each  $j = 0, 1, 2, \dots$

That is  $A^{i_j}(i_j^* \pi^j) = A\pi^j$ , where  $\pi^j : X \rightarrow X_j i_j$  is an  $\mathbf{R}$  linear projection with  $\pi^j(x) = x_j i_j$  so that  $\sum_j \pi^j = I$ .

It can be lightly seen, that Definition 1 is natural. Indeed, if  $\mathbf{A}$  is a quasi-commutative algebra over the Cayley-Dickson algebra  $\mathcal{A}_v$  with  $2 \leq v$ , then an algebra  $\mathbf{A}_0 i_0 \oplus \mathbf{A}_k i_k$  is commutative for  $k \geq 1$  over the complex field  $\mathbf{C}_{i_k} := \mathbf{R} \oplus \mathbf{R} i_k$ , since there is the decomposition  $\mathbf{A} = \mathbf{A}_0 i_0 \oplus \mathbf{A}_1 i_1 \oplus \dots \mathbf{A}_m i_m \oplus \dots$  with pairwise isomorphic real algebras  $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_m, \dots$ .

Let  $\mathcal{B}(Y, \mathcal{A}_v)$  denote the algebra of all bounded Borel functions from a topological space  $Y$  into the Cayley-Dickson algebra  $\mathcal{A}_v$ , let also  $\mathcal{B}_u(Y, \mathcal{A}_v)$  denote the algebra of all Borel functions from  $Y$  into  $\mathcal{A}_v$  with point-wise addition and multiplication of functions and multiplication of functions  $f$  on the left and on the right on Cayley-Dickson numbers  $a, b \in \mathcal{A}_v$ ,  $2 \leq v$ .

**3. Lemma.** *If  $T$  is a closed symmetrical operator in a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ , then ranges  $\mathcal{R}(T \pm MI)$  of  $(T \pm MI)$  are closed for each  $M \in \mathcal{S}_v$ , when  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ . If  $T$  is closed and  $0 \leq \langle Tz; z \rangle$  for each  $z \in \mathcal{D}(T)$ , then  $T + I$  has a closed range.*

**Proof.** Let  ${}_n x$  be a sequence in  $\mathcal{D}(T)$  so that  $\{(T \pm MI) {}_n x : n\}$  tends to a vector  $y \in X$ . But  $\langle Tz; z \rangle$  is real for each  $z \in \mathcal{D}(T)$ , hence

$$\begin{aligned} \|z\|^2 &\leq (\langle Tz; z \rangle^2 + \langle z; z \rangle^2)^{1/2} = | \langle (T \pm MI)z; z \rangle | \\ &\leq \| (T \pm MI)z \| \|z\|, \end{aligned}$$

consequently,  $\|{}_n x - {}_m x\| \leq \| (T \pm MI)({}_n x - {}_m x) \|$  and  ${}_n x$  converges to some vector  $x \in X$ . On the other hand, the sequence  $\{T {}_n x : n\}$  converges to  $\mp Mx + y$  and the operator  $T$  is closed, consequently,  $x \in \mathcal{D}(T)$  and  $Tx = \mp Mx + y$ . Indeed,

$$M^*(Mx) = M^* \sum_j x_j (Mi_j) = \sum_j x_j i_j = x,$$

since  $\|Mx\| = \|x\|$  and  $M(x_j i_j) = x_j(Mi_j)$  and  $M^*(Mi_j) = i_j$  for each  $x_j \in X_j$  and  $M \in \mathcal{S}_v$  for  $2 \leq v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ . Thus  $(T \pm MI)x = y$ , hence the operators  $(T \pm MI)$  have closed ranges for such  $M$ .

If an operator  $T$  is closed and  $0 \leq \langle Tz; z \rangle$  for each  $z \in \mathcal{D}(T)$ , then

$$\|z\|^2 \leq \langle z; z \rangle + \langle Tz; z \rangle \leq \|(T + I)z\| \|z\|$$

and analogously to the proof above we get that  $(T + I)$  has a closed range.

**4. Proposition.** *If  $T$  is a closed symmetrical operator on a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ , then the following statements are equivalent:*

- (1) *an operator  $T$  is self-adjoint;*
- (2) *operators  $(T^* \pm MI)$  have  $\{0\}$  as null space for each  $M \in \mathcal{S}_v$ , when  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ ;*
- (3) *operators  $(T \pm MI)$  have  $X$  as range for every  $M \in \mathcal{S}_v$ , when  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ ;*
- (4) *operators  $(T \pm MI)$  have ranges dense in  $X$  for all  $M \in \mathcal{S}_v$  with  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ .*

**Proof.** (1)  $\Rightarrow$  (2). We have  $\langle Tx; x \rangle = \langle x; Tx \rangle \in \mathbf{R}$  for each  $x \in \mathcal{D}(T)$ , when  $T^* = T$ , hence

$$\langle (T^* \pm MI)x; x \rangle = \langle (T \pm MI)x; x \rangle = \langle Tx; x \rangle \pm M\|x\|^2,$$

consequently,  $\langle (T^* \pm MI)x; x \rangle = 0$  only when  $x = 0$ , since  $\|Mx\| = \|x\|$  for each  $M \in \mathcal{S}_v$  with  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ . Thus each operator  $(T^* \pm MI)$  has  $\{0\}$  as null space.

(2)  $\Rightarrow$  (3). Ranges  $\mathcal{R}(T \pm MI)$  are closed due to Lemma 3. Therefore, it is sufficient to show that these ranges are dense in a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ . But  $\langle Tx; y \rangle = \mp M \langle x; y \rangle$ , when  $\langle (T \pm MI)x; y \rangle = 0$  for all  $x \in \mathcal{D}(T)$ , consequently,  $y \in \mathcal{D}(T^*)$  and  $T^*y = \pm My$ . Therefore,  $y = 0$ , since the operators  $T^* \pm MI$  have  $\{0\}$  as null space. Thus the operators  $(T \pm MI)$  have dense ranges for each  $M \in \mathcal{S}_v$  when  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ .

(3)  $\Leftrightarrow$  (4). This follows from the preceding demonstrations.

(3)  $\Rightarrow$  (1). When  $T$  is a closed and symmetrical operator, one has that  $T \subseteq T^*$  and a graph  $\Gamma(T)$  is a closed subspace of the closed  $\mathbf{R}$ -linear space  $\Gamma(T^*)$ . The equality

$$\langle y; x \rangle + \langle T^*y; Tx \rangle = 0$$

is valid for each  $x \in \mathcal{D}(T)$ , when  $(y, T^*y) \in \Gamma(T^*)$  is orthogonal to  $\Gamma(T)$ . Operators  $(T \pm MI)$  have range  $X$ , consequently, there exists a vector  $x \in \mathcal{D}(T)$  so that  $Tx \in \mathcal{D}(T)$  and  $y = (T + MI)(T - MI)x = (T^2 + I)x$ , since  $T \subseteq T^*$  and  $T(MI) \subseteq (MI)T^*$ . For such vector  $x$  one gets

$$\langle y; y \rangle = \langle y; (T^2 + I)x \rangle = \langle y; x \rangle + \langle T^*y; Tx \rangle = 0,$$

hence  $\langle y; T^*y \rangle = (0, 0)$  and  $\Gamma(T) = \Gamma(T^*)$  and hence  $T^* = T$ . That is, the operator  $T$  is self-adjoint.

**5. Note.** If an operator  $T$  in a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ , is self-adjoint, the fact that operators  $(T \pm MI)$  have dense everywhere defined bounded inverses with bound not exceeding one follows from Proposition 4 and the inequality at the beginning of the demonstration of Lemma 3 for  $M \in \mathcal{S}_v$  with  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ .

For an operator  $T$  let  $\text{alg}_{\mathcal{A}_v}(I, T) =: \mathbf{Q}$  be a family of operators generated by  $I$  and  $T$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ . Consider this family of operators on a common domain  $\mathcal{D}^\infty(T) := \cap_{n=1}^\infty \mathcal{D}(T^n)$ . Then the family  $\mathbf{Q}$  on  $\mathcal{D}^\infty(T)$  can be considered as an  $\mathcal{A}_v$  vector space. Take the decomposition  $\mathbf{Q} = \mathbf{Q}_0 i_0 \oplus \mathbf{Q}_1 i_1 \oplus \dots \oplus \mathbf{Q}_m i_m \oplus \dots$  of this  $\mathcal{A}_v$  vector space with pairwise isomorphic real vector spaces  $\mathbf{Q}_0, \mathbf{Q}_1, \dots, \mathbf{Q}_m, \dots$  and for each operator  $B \in \mathbf{Q}$  put

$$(1) \quad B = \sum_j {}^j B \text{ with } {}^j B = \hat{\pi}^j(B) \in \mathbf{Q}_j i_j$$

for each  $j$ , where  $\hat{\pi}^j : \mathbf{Q} \rightarrow \mathbf{Q}_j i_j$  is the natural  $\mathbf{R}$  linear projection, real linear spaces  $\mathbf{Q}_m i_m$  and  $i_m \mathbf{Q}_m$  are considered as isomorphic. Evidently, there is the inclusion of domains of these  $\mathbf{R}$  linear operators  $\mathcal{D}({}^k B) \supset \mathcal{D}(B)$  for each  $k$ , particularly,  $\mathcal{D}({}^k T) \supset \mathcal{D}(T)$ .

**6. Lemma.** *Let  $\{ {}_n E : n \}$  be an increasing sequence of  $\mathcal{A}_v$  graded projections on a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$  and let also  $G$  be an  $\mathbf{R}$  homogeneous  $\mathcal{A}_r$  additive operator with dense domain  $\bigcup_n {}_n E(X) =: \mathcal{E}$  such that  $G {}_n E$  is a bounded self-adjoint operator on  $X$ , where  $2 \leq v$ . Then  $G$  is pre-closed and its closure  $T$  is self-adjoint. Moreover, if an operator  $T$  is closed with core  $\mathcal{E}$  and  $T {}_n E$  is a bounded self-adjoint operator for each  $n \in \mathbf{N}$ , then  $T$  is self-adjoint.*

**Proof.** For each  $x, y \in \mathcal{E}$  there exists a natural number  $m$  so that

$$\langle Gx; y \rangle = \langle G_m E x; y \rangle = \langle x; G_m E y \rangle = \langle x; G y \rangle,$$

hence  $y \in \mathcal{D}(G^*)$  and  $G^*$  is densely defined so that  $G$  is pre-closed.

Consider now the closure  $T$  of  $G$ . For each  $x, y \in \mathcal{D}(T)$  there exist sequences  ${}_n x$  and  ${}_n y$  in  $\mathcal{E}$  for which  $\lim_n {}_n x = x$ ,  $\lim_n {}_n y = y$ ,  $\lim_n T {}_n x = Tx$  and  $\lim_n T {}_n y = Ty$ . Therefore, the equalities

$$\langle T {}_n x; {}_n y \rangle = \langle T {}_m E {}_n x; {}_n y \rangle = \langle {}_n x; T {}_m E {}_n y \rangle = \langle {}_n x; T {}_n y \rangle$$

are satisfied, consequently,

$$\lim_n \langle T {}_n x; {}_n y \rangle = \langle Tx; y \rangle \text{ and}$$

$$\lim_n \langle {}_n x; T {}_n y \rangle = \langle x; Ty \rangle.$$

Thus one gets  $\langle Tx; y \rangle = \langle x; Ty \rangle$  and that the operator  $T$  is symmetric. Mention that  $(T \pm MI) {}_n E(X) = {}_n E(X)$  for each  $M \in \mathcal{S}_v$  with  $v \leq 3$  or  $M \in \{i_1, i_2, \dots\}$ , since  $T {}_n E$  is bounded and self-adjoint, for which the operator  $T {}_n E \pm M {}_n E$  has a bounded inverse on  ${}_n E(X)$ . This implies that  $T \pm MI$  has a dense range and it coincides with  $X$ , consequently,  $T$  is self-adjoint due to Lemma 3 and Proposition 4.

**7. Theorem.** Let  $\mu$  be a  $\sigma$ -finite measure  $\mu : \mathcal{F} \rightarrow [0, \infty]$  on a  $\sigma$ -algebra  $\mathcal{F}$  of subsets of a set  $S$  and let  $L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$  be a Hilbert space completion of the set of all step  $\mu$  measurable functions  $f : S \rightarrow \mathcal{A}_v$  with the  $\mathcal{A}_v$  valued scalar product

$$\langle f; g \rangle = \int f(x) \tilde{g}(x) \mu(dx)$$

for each  $f, g \in L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$ , where  $2 \leq v$ . Suppose that  $\mathbf{A}$  is its left multiplication algebra  $M_g f = gf$ . Then an  $\mathbf{R}$ -linear  $\mathcal{A}_v$ -additive operator  $T$  is affiliated with  $\mathbf{A}$  if and only if a measurable finite  $\mu$  almost everywhere on  $S$  function  $g : S \rightarrow \mathcal{A}_v$  exists so that  $T = M_g$  and  $g_k(t) f_j(t) = (-1)^{\kappa(j,k)} f_j(t) g_k(t)$  for  $\mu$  almost all  $t \in S$  and each  $f \in L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$  for each  $j, k = 0, 1, \dots$ , where  $\kappa(j, k) = 0$  for  $j = 0$  or  $k = 0$  or  $j = k$ , while  $\kappa(j, k) = 1$  for each  $j \neq k \geq 1$ . Moreover, an operator  $T \eta \mathbf{A}$  is self-adjoint if and only if  $g$  is real-valued  $\mu$  almost everywhere on  $S$ .

**Proof.** If  $g : S \rightarrow \mathcal{A}_v$  is a  $\mu$  measurable  $\mu$  essentially bounded on  $S$  function, then  $M_g$  is a bounded  $\mathbf{R}$ -linear  $\mathcal{A}_v$ -additive operator on  $L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$ , since

$\|M_g f\|_2 \leq \|g\|_\infty \|f\|_2$ , where

$\|g\|_\infty = \text{ess sup}_{x \in S} |g(x)|$  and  $\|f\|_2 := \sqrt{\langle f; f \rangle}$ . That is  $M_g \in \mathbf{A}$ .

Each operator  $G \in \mathbf{A}$  is an  $(\mathcal{A}_v)_{\mathbf{C}_i}$  combination of unitary operators in  $\mathbf{A}$  (see Theorem II.2.20 [28]) and  $UT \subseteq TU$  for each unitary operator  $U \in \mathbf{A}$ , since  $T\eta\mathbf{A}$  and  $\mathbf{A} \subseteq \mathbf{A}^*$ , where  $\mathbf{C}_i = \mathbf{R} \oplus \mathbf{R}i$ . If  $F$  is an  $\mathcal{A}_v$  graded projection operator corresponding to the characteristic function  $\chi_P$  of a  $\mu$ -measurable subset  $P$  in  $S$ , this implies the inclusion  $FT \subseteq TF$ , consequently,  $Ff \in \mathcal{D}(T)$  for each  $f \in \mathcal{D}(T)$ . If  $f \in \mathcal{D}(T)$  and  ${}_n F$  corresponds to the multiplication by the characteristic function  $\chi_K$  of the set  $K = \{x \in S : |f(x)| \leq n\}$ , then  ${}_n F$  is an ascending sequence of projections in the algebra  $\mathbf{A}$  converging to the unit operator  $I$  relative to the strong operator topology, since  $f$  is finite almost everywhere,  $\mu\{x : |f(x)| = \infty\} = 0$ . Therefore,  ${}_n Ff \in \mathcal{E}$  for each  $n$ , where  $\mathcal{E}$  denotes the set of all  $\mu$  essentially bounded functions  $f \in \mathcal{D}(T)$ . Moreover, the limits exist:

$$\lim_n {}_n Ff = f \text{ and } \lim_n T {}_n Ff = \lim_n {}_n F T f = T f.$$

Thus  $\mathcal{E} = \bigcup_n {}_n F \mathcal{D}(T)$ , where  $\bigcup_n {}_n F \mathcal{D}(T)$  is dense in  $\mathcal{D}(T)$ . Thus  $\mathcal{E}$  is a core of  $T$ .

Each step function  $u : S \rightarrow \mathcal{A}_v$  on  $(S, \mathcal{F})$  has the form

$$u(s) = \sum_{l=1}^m c_l \chi_{B_l},$$

where  $B_l \in \mathcal{F}$  and  $c_l \in \mathcal{A}_v$  for each  $l = 1, \dots, m$ ,  $m \in \mathbf{N}$ , where  $\chi_B$  denotes the characteristic function of a subset  $B$  in  $S$  so that  $\chi_B(s) = 1$  for each  $s \in B$  and  $\chi_B(s) = 0$  for all  $s$  outside  $B$ . A subset  $N$  in  $S$  is called  $\mu$  null if there exists  $H \in \mathcal{F}$  so that  $N \subset H$  and  $\mu(H) = 0$ . If consider an algebra  $\mathcal{F}_\mu$  of subsets in  $S$  which is the completion of  $\mathcal{F}$  by  $\mu$  null subsets, then each step function in  $L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$  may have  $B_l \in \mathcal{F}_\mu$  for each  $l$ .

For each functions  $f, g \in \mathcal{E}$  there are the equalities

$$\begin{aligned} (1) \quad & ((f_j i_j) {}^k T) g_k = (M_{f_j i_j} {}^k T) g_k = (-1)^{\kappa(j,k)} ({}^k T M_{f_j i_j}) g_k = \pm {}^l T f_j g_k \\ & = (-1)^{\kappa(j,k)} M_{g_k i_k} {}^j T f_j = (-1)^{\kappa(j,k)} g_k i_k [{}^j T f_j], \end{aligned}$$

where  $i_j i_k = \pm i_l$ ,  $\kappa(j, k) = 0$  for  $j = 0$  or  $k = 0$  or  $j = k$ , while  $\kappa(j, k) = 1$  for each  $j \neq k \geq 1$ ,

$$f = \sum_j f_j i_j$$

with real-valued functions  $f_j$  for each  $j$ . Let  $S_k \in \mathcal{F}$  be a sequence of pairwise disjoint subsets with  $0 < \mu(S_k) < \infty$  for each  $k$  and with union  $\bigcup_k S_k = S$ . For the characteristic function  $\chi_{S_k}$  a sequence  $\{f^{k,j} : j \in \mathbf{N}\} \subset \mathcal{E}$  of real-valued functions exists converging to  $\chi_{S_k}$  in  $L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$ . The set  $S_k^0 := \{t \in S_k : f^{k,j}(s) = 0 \ \forall j\}$  has  $\mu$  measure zero, since

$$0 = \lim_j M_g f^{k,j} = M_g \chi_{S_k} = g, \text{ where } g = \chi_{S_k^0}.$$

Put  $h(s) = [(T f^{k,j})(s)][f^{k,j}(s)]^{-1}$  for  $s \in S_k \setminus S_k^0$ , where  $j$  is the least natural number so that  $f^{k,j}(s) \neq 0$ . Thus  $h$  is a measurable function defined  $\mu$  almost everywhere on  $S$ .

In accordance with Formula (1) the equality

$$(2) \quad f^{k,j}(s)[Tf]_l(s) = \sum_{p,q; i_q i_p = i_l} \{(-1)^{\kappa(q,p)} f_p(s) i_p [T f^{k,j}]_q(s) i_q + f_q(s) i_q [T f^{k,j}]_p(s) i_p\}$$

is accomplished for all  $k, j \in \mathbf{N}$  and  $l = 0, 1, 2, \dots$  except for a set of measure zero, consequently,  $[Tf](s) = h(s)f(s) = M_h f(s)$  almost everywhere on  $S$  so that  ${}^k T f = [Tf]_k i_k$  for each  $k$ .

The operator  $M_h$  is closed and affiliated with  $\mathbf{A}$  as follows from the demonstration above. On the other hand,  $M_h$  is an extension of  $T|_{\mathcal{E}}$ , consequently,  $T \subseteq M_h$ .

The family of all functions  $z \in L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$  vanishing on  $\{s \in S : |h(s)| > m\}$  for some  $m = m(z) \in \mathbf{N}$  forms a core for  $M_h$ . For such a function  $z$  take a sequence  $f^k$  of functions in  $\mathcal{E}$  tending to  $z$  in  $L^2(S, \mathcal{F}, \mu, \mathcal{A}_v)$ . Evidently  $f^k$  can be replaced by  $y f^k$ , where  $y$  is the characteristic function of the set  $\{s \in S : z(s) \neq 0\}$  of all points at which  $z$  does not vanish. Thus we can choose a sequence  $f^k$  vanishing for each  $k$  when  $z$  does. Therefore,

$$(3) \quad \lim_k T f^k = \lim_k M_h M_y f^k = M_h M_y z = M_h z,$$

since the operator  $M_h M_y$  is bounded.

For a closed operator  $T$  this implies that  $z \in \mathcal{D}(T)$  and  $Tz = M_h z$ , consequently,  $T = M_h$ .

If an operator  $T$  is self-adjoint, then  $M_{yh}$  is a bounded self-adjoint operator, hence the function  $yh$  is real-valued  $\mu$  almost everywhere on  $S$ .

For a bounded multiplication operator  $M_g$  the  $\mathcal{A}_v$  graded projections  ${}_t E$  corresponding to multiplication by characteristic function of the set  $\{s \in$



$S : g(s) \leq t\}$  forms a spectral  $\mathcal{A}_v$  graded spectral resolution  $\{tE : t \in \mathbf{R}\}$  of the identity for the operator  $T$  (see also Theorem 2.28 [20]).

**8. Definition.** Suppose that  $V$  is an extremely disconnected compact Hausdorff topological space and  $V \setminus W$  is an open dense subset in  $V$ . If a function  $f : V \setminus W \rightarrow \mathcal{A}_v$  is continuous and

$$\lim_{x \rightarrow y} |f(x)| = \infty$$

for each  $y \in W$ , where  $x \in V \setminus W$ ,  $1 \leq v$ , then  $f$  is called a normal function on  $V$ .

If a normal function on  $V$  is real-valued it will be called a self-adjoint function on  $V$ .

The families of all normal and self-adjoint functions on  $V$  we denote by  $\mathcal{N}(V, \mathcal{A}_v)$  and  $\mathcal{Q}(V)$  respectively, let also

$$W_+ := W_+(f) := \{y \in W : \lim_{x \rightarrow y} f(x) = \infty\},$$

$$W_- := W_-(f) := \{y \in W : \lim_{x \rightarrow y} f(x) = -\infty\}$$

for a self-adjoint function  $f$  on  $V$ .

**9. Lemma.** Let  $f$  and  $g$  be normal functions on  $V$  (see Definition 8) defined on  $V \setminus W_f$  and  $V \setminus W_g$  respectively so that  $f(x) = g(x)$  for each  $x$  in a dense subset  $U$  in  $V \setminus (W_f \cup W_g)$ . Then  $W_f = W_g$  and  $f = g$ .

**Proof.** The subset  $V \setminus (W_f \cup W_g)$  is dense in  $V$ , consequently,  $U$  is dense in  $V$ . If  $y \in W_g$ , then for each  $N > 0$  there exists a neighborhood  $E$  of  $y$  in  $V$  so that  $|g(x)| > N$  for each  $x \in E \cap (V \setminus W_g)$ , hence  $|f(x)| = |g(x)| > N$  for each  $x \in E \cap (V \setminus (W_g \cup W_f))$ , consequently,  $W_g \subset W_f$  and symmetrically  $W_f \subset W_g$ . Thus  $W_f = W_g$  and  $f - g$  is defined and continuous on  $V \setminus W_f$  and is zero on  $U$ , hence  $f = g$ .

**10. Lemma.** Suppose that  $T$  is a self-adjoint operator acting on a Hilbert space  $X$  over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ , so that  $T$  is affiliated with some quasi-commutative von Neumann algebra  $\mathbf{A}$  over  $\mathcal{A}_v$ , where  $\mathbf{A}$  is isomorphic to  $C(\Lambda, \mathcal{A}_v)$  with an extremely disconnected compact Hausdorff topological space  $\Lambda$ . Then there is a unique self-adjoint function  $h$  on  $\Lambda$  such that  $h \hat{\cdot} e \in C(\Lambda, \mathcal{A}_v)$  and a function  $h \hat{\cdot} e$  represents  $TE$ , when  $E$  is an  $\mathcal{A}_v$  graded projection for which  $TE \in L_q(X)$

is a bounded operator on  $X$ , a function  $e \in C(\Lambda, \mathcal{A}_v)$  corresponds to  $E$ ,  $h \cdot e(x) = h(x)$  for  $e(x) = 1$ , while  $h \cdot e(x) = 0$  otherwise. There exists an  $\mathcal{A}_v$  graded resolution of the identity  $\{E_b : b\}$  so that  $\bigcup_{n=1}^{\infty} {}_nF(X)$  is a core for  $T$ , where  ${}_nF := {}_nE - {}_{-n}E$  and

$$(1) \quad Tx = \int_{-n}^n d_b E_b x$$

for every  $x \in {}_nF(X)$  and each  $n$  in the sense of norm convergence of Riemann sums.

**Proof.** Take  $Y = X \oplus \mathbf{i}X$ , where  $\mathbf{i}$  is a generator commuting with  $i_j$  for each  $j$  such that  $\mathbf{i}^2 = -1$ . Take an extension  $T$  onto  $\mathcal{D}(T) \oplus \mathcal{D}(T)\mathbf{i}$  so that  $T(x + y\mathbf{i}) = Tx + (Ty)\mathbf{i}$  for each  $x, y \in \mathcal{D}(T)$ . Then  $\mathcal{R}(T + \mathbf{i}I) = Y$  and  $\mathcal{R}(T - \mathbf{i}I) = Y$ , where  $\mathcal{R}(T \pm \mathbf{i}I) = (T \pm \mathbf{i}I)Y$ ,  $\ker(T \pm \mathbf{i}I) = \{0\}$  and inverses  $B_{\pm} := (T \pm \mathbf{i}I)^{-1}$  are everywhere defined on  $Y$  and of norm not exceeding one in accordance with §II.2.74 and Proposition II.2.75 [28]. Then the equalities

$$\langle B_+(T + \mathbf{i}I)x; (T - \mathbf{i}I)y \rangle = \langle x; (T - \mathbf{i}I)y \rangle = \langle (T + \mathbf{i}I)x; y \rangle = \langle (T + \mathbf{i}I)x, B_-(T - \mathbf{i}I)y \rangle$$

are satisfied for each  $x$  and  $y \in \mathcal{D}(T)$ , since the operator  $T$  is self-adjoint, hence  $B_- = B_+^*$ .

Then an arbitrary vector  $z \in X$  has the form  $z = B_-B_+x$  for the corresponding vector  $x \in \mathcal{D}(T)$  with  $Tx \in \mathcal{D}(T)$ , since  $\mathcal{R}(B_{\pm}) = Y$ . In the latter case we get  $B_-B_+x = (TT - \mathbf{i}IT + T\mathbf{i}I + I)x = B_+B_-x$ , since to  ${}_nF$  a real-valued function  ${}_nf \in C(\Lambda, \mathbf{R})$  corresponds for each  $n$ . On the other hand,  $B_- = B_+^*$ , consequently, the operator  $B_+$  is normal.

Consider a quasi-commutative von Neumann algebra  $\check{\mathbf{A}}$  over the complexified algebra  $(\mathcal{A}_v)_{\mathbf{C}_i}$  containing  $I$ ,  $B_+$  and  $B_- \in \mathbf{A}$ . If  $U$  is a unitary operator in  $\mathbf{A}^*$  (see §II.2.71 [28]), then

$$Ux = UB_+(T + \mathbf{i}I)x = (B_+)U(T + \mathbf{i}I)x \text{ hence}$$

$$(T + \mathbf{i}I)Ux = U(T + \mathbf{i}I)x,$$

consequently,  $T$  is affiliated with  $\check{\mathbf{A}}$  by Definition 1, since  $(\mathbf{i}I)Ux = U(\mathbf{i}I)x$  for a unitary operator  $U$  as follows from the definition of a unitary operator and the  $\mathcal{A}_v$  valued scalar product on  $X$  extended to the  $(\mathcal{A}_v)_{\mathbf{C}_i}$  valued scalar product on  $Y$ :

$$\langle a + b\mathbf{i}; c + q\mathbf{i} \rangle = (\langle a; c \rangle + \langle b; q \rangle) + (\langle b; c \rangle - \langle a; q \rangle)\mathbf{i}$$

for each vectors  $a, b, c, q \in X$ .

The algebra  $\check{\mathbf{A}}$  has the decomposition  $\check{\mathbf{A}} = \mathbf{A}^0 \oplus \mathbf{A}^1\mathbf{i}$ , where  $\mathbf{A}^0$  and  $\mathbf{A}^1$  are quasi-commutative isomorphic algebras over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ .

In view of Theorem 2.24 [20]  $\check{\mathbf{A}}$  is isomorphic with  $C(\Lambda, (\mathcal{A}_v)_{\mathbf{C}_i})$  for some extremely disconnected compact Hausdorff topological space  $\Lambda$ , since the generator  $\mathbf{i}$  has the real matrix representation  $\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and the real field is the center of the algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ . Let  $f_+$  and  $f_-$  be functions corresponding to  $B_+$  and  $B_-$  respectively. Put  $h_{\pm} = 1/f_{\pm}$  at those points where  $f_{\pm}$  does not vanish. Therefore, these functions  $h_{\pm}$  are continuous on their domains of definitions as well as  $h = \frac{1}{2}(h_+ + h_-)$ . The function  $h$  corresponds to  $T$ .

It will be demonstrated below that  $h$  is real-valued and then the  $\mathcal{A}_v$  graded spectral resolution of  $T$  will be constructed.

At first it is easy to mention that  $f_+^* = f_-$ , since  $B_+^* = B_-$ , consequently,  $h$  is real-valued. The functions  $f_+$  and  $f_-$  are continuous and conjugated, hence the set  $W := f_+^{-1}(0) = f_-^{-1}(0)$  is closed. If the set  $W$  contains some non-void open subset  $J$ , then  $cl(J) \subset W$  so that  $cl(J)$  is clopen (i.e. closed and open simultaneously) in  $\Lambda$ , consequently,  $W$  is nowhere dense in  $\Lambda$ . The projection  $P$  corresponding to this subset  $cl(J)$  would have the zero product  $B_+P = 0$  contradicting the fact that  $ker(B_+) = \{0\}$ . Thus each point  $x \in W$  is a limit point of points  $y \in \Lambda \setminus W$ , but  $h_+$  and  $h_-$  are defined on the latter set. Therefore, the set  $\Lambda \setminus W$  is dense in  $\Lambda$  and the functions  $h_+$  and  $h_-$  are defined on  $\Lambda \setminus W$ .

Then  $TB_+B_-y = (T + \mathbf{i}I - \mathbf{i}I)B_+B_-y = B_-y - (\mathbf{i}I)B_+B_-y$  for each  $y \in \Lambda$ , hence  $TB_+B_- = B_- - (\mathbf{i}I)B_+B_-$  and analogously  $TB_-B_+ = B_+ + (\mathbf{i}I)B_-B_+$  and inevitably

$$(2) \quad 2(\mathbf{i}I)B_+B_- = B_- - B_+ \text{ and}$$

$$(3) \quad TB_+B_- = \frac{1}{2}(B_+ + B_-).$$

From Formula (2) and the definitions of functions  $f_{\pm}$  and  $h$  one gets  $(h(y) + \mathbf{i})^{-1} = f_+(y)$  and  $(h(y) - \mathbf{i})^{-1} = f_-(y)$  for each  $y \in \Lambda \setminus W$ , since  $(h(y) + \mathbf{i})f_+(y) = \frac{1}{2} + \frac{1}{2f_-}f_+ + \mathbf{i}f_+ = \frac{1}{2} + \frac{w-\mathbf{i}}{2} \frac{1}{w+\mathbf{i}} + \mathbf{i} \frac{1}{w+\mathbf{i}} = 1 - \frac{2\mathbf{i}}{2} \frac{1}{w+\mathbf{i}} + \mathbf{i} \frac{1}{w+\mathbf{i}} = 1$ ,

where  $w \pm \mathbf{i} = \frac{1}{f_{\pm}}$  corresponds to  $(T \pm \mathbf{i}I)$ . But  $f_+(y)$  tends to zero when  $y \in \Lambda \setminus W$  tends to  $x \in W$ , consequently,  $\lim_{y \rightarrow x} |h(y)| = \infty$ . This means that  $h$  is a self-adjoint function on  $\Lambda$ .

We put  $U_b := \{x \in \Lambda \setminus W : h(x) > b\}$  and  $V_b = U_b \cup W_+(h)$  for  $b \in \mathbf{R}$  (see Definition 8). The function  $h$  is continuous on the open subset  $\Lambda \setminus W$  in  $\Lambda$ , hence the subset  $U_b$  is open in  $\Lambda$ . For a point  $x \in W_+$  there exists an open set  $Q$  containing  $x$  such that  $h(y) > \max(b, 0)$  for each  $y \in Q \cap (\Lambda \setminus W)$ , since  $\lim_{y \rightarrow x} |h(y)| = \infty$ , where  $W_+ = W_+(h)$  (see Definition 8). Therefore, this implies the inclusion  $Q \cap (\Lambda \setminus W) \subset U_b \subset V_b$ .

Suppose that there would exist a point  $z \in Q \cap W_-$ , where  $W_- = W_-(h)$ , then a point  $y \in Q$  with  $h(y) < 0$  would exist contradicting the choice of  $Q$ . This means that  $Q \cap W \subset W_+$  and  $Q \subset V_b$ , consequently,  $V_b$  is open in  $\Lambda$ .

We consider next the subset  $\Lambda_b := \Lambda \setminus cl(V_b)$ . In accordance with §2.24 [20] the set  $\Lambda_b$  contains every clopen subset  $K_b := \{y \in \Lambda : h(y) \leq b\}$ , where  $h(y) = -\infty$  for  $y \in W_-$  so that  $W_- \subset K_b$  for each  $b \in \mathbf{R}$ . To demonstrate this suppose that  $y \in V_b$ , then  $y \in \Lambda \setminus K_b$  and  $cl(V_b) \subset \Lambda \setminus K_b$  and  $K_b \subset \Lambda \setminus cl(V_b) = \Lambda_b$ , since  $\Lambda \setminus K_b$  is closed. Moreover, we have  $h(y) \leq b$  for each  $y \in \Lambda_b \cap (\Lambda \setminus W)$ , since  $y \notin U_b$ , while the set  $\Lambda_b$  is clopen in  $\Lambda$ . Therefore,  $y \in W \setminus W_+$  and  $h(y) \leq b$  for each  $y \in \Lambda_b \cap W$  and  $W_- = W \setminus W_+$ . Thus  $\Lambda_b$  is the largest clopen subset in  $\Lambda$  so that  $h(y) \leq b$  and  $\Lambda_b \cap W = W_-$ .

Denote by  $e_b$  the characteristic function of the subset  $\Lambda_b$  and  ${}_bE$  be an  $\mathcal{A}_v$  graded projection operator in  $\mathbf{A}$  corresponding to  $e_b$ , where  $b \in \mathbf{R}$  (see also §2.24 [20]). The subset  $W$  is nowhere dense in  $\Lambda$ , consequently,  $\vee_b e_b = 1$  and  $\wedge_b e_b = 0$  such that  $\vee_b {}_bE = 1$  and  $\wedge_b {}_bE = 0$ . Thus  $\{{}_bE : b\}$  is the  $\mathcal{A}_v$  graded resolution of the identity so that  ${}_bE_s = (sI) {}_bE = {}_bE(sI)$  corresponds to  $e_b s = s e_b$  for each marked Cayley-Dickson number  $s \in \mathcal{A}_v$ . This resolution of the identity is unbounded when  $h \notin C(\Lambda, \mathcal{A}_v)$ .

Put  $F = {}_bE - {}_aE$  for  $a < b \in \mathbf{R}$ , hence  $e_b - e_a =: u$  is the characteristic function of  $\Lambda_b \setminus \Lambda_a$  corresponding to  $F$ . Therefore, the inclusion  $\Lambda_b \setminus \Lambda_a \subset \Lambda \setminus W$  follows and  $f_+(y)f_-(y) \neq 0$  when  $u(y) = 1$ , since  $\Lambda_b \cap W = \Lambda_a \cap W = W_-$ . Then

$$(4) \quad h(y) = \frac{f_+ + f_-}{2f_+f_-}(y)$$

for each  $y \in \Lambda \setminus W$ . The function  $f_+f_-$  is continuous and vanishes nowhere on the clopen subset  $\Lambda_b \setminus \Lambda_a$ , consequently, a positive continuous function  $\psi$  on  $\Lambda$  exists so that  $\psi f_+f_- = u$  and  $\psi u = \psi$ . Consider an element  $\Psi$  of the algebra  $\mathbf{A}$  corresponding to  $\psi$ , hence

$$(5) \quad \Psi B_+ B_- = F.$$

On the other hand, from the construction above it follows that  $a \leq h(y) \leq b$  for each  $y \in \Lambda_b \setminus \Lambda_a$  and from Formula (4) one gets

$$\begin{aligned} a f_+ f_- u &\leq \frac{(f_+ + f_-)u}{2} \leq b f_+ f_- u \text{ and} \\ a \psi f_+ f_- u &= a u \leq \frac{(f_+ + f_-)\psi u}{2} = \frac{(f_+ + f_-)\psi}{2} \leq b \psi f_+ f_- u = b u, \end{aligned}$$

since the real field is the center of the Cayley-Dickson algebra  $\mathcal{A}_v$ . Thus

$$(6) \quad aF \leq \frac{(B_+ + B_-)\Psi}{2} \leq bF.$$

Therefore, Formulas (3, 5, 6) imply that

$$(7) \quad aF \leq TF \leq bF.$$

Therefore, the operator  $TF$  is bounded and the element  $h\hat{u} \in C(\Lambda, \mathcal{A}_v)$  corresponds to it due to (3 – 5). If  $E$  is an  $\mathcal{A}_v$  graded projection belonging to  $\mathbf{A}$  so that  $TE \in L_q(X)$  and  $U$  is a unitary operator in  $\mathbf{A}^*$  such that  $U^{-1}TEU = U^{-1}TUE = TE$  one has  $TE \in \mathbf{A}$  (see §II.2.71 [28]). Let each function  ${}_ng \in C(\Lambda, \mathcal{A}_v)$  be corresponding to the  $\mathcal{A}_v$  graded projection  ${}_nF$  and  $\Lambda_n := {}_ng^{-1}(1)$ . Then  $\bigcup_n \Lambda_n$  is dense in  $\Lambda$ , since  $\bigvee_n {}_nF = I$ . If a function  $e \in C(\Lambda, \mathcal{A}_v)$  corresponds to  $E$ , then  $(h\hat{e}){}_ng$  corresponds to the operator  $T {}_nFE$ , where  ${}_nFE = E {}_nF$ .

Suppose that  $e(y) = 1$  for some  $y \in \Lambda$ . For each (open) neighborhood  $H$  of  $y$  there exist  $n \in \mathbf{N}$  and  $x \in \Lambda_n$  such that  $x \in H$ , hence  $((h\hat{e}){}_ng)(x) = h(x)$  and  $|h(x)| \leq \|TE {}_nF\| \leq \|TE\|$ . Thus  $y \notin W$  and  $|h(y)| \leq \|TE\|$  and hence  $h\hat{e} \in C(\Lambda, \mathcal{A}_v)$ . Then one gets also  $(h_j\hat{e}){}_ng_k e_l = h_j\hat{e}({}_ng_k e_l) = (h_j\hat{e}){}_ng_k$  for each  $j, k, l$ . If  $s \in C(\Lambda, \mathcal{A}_v)$  represents  $TE$ , then  $s {}_ng$  represents  $TE {}_nF$ , that is  $s {}_ng = (h\hat{e}){}_ng$  for every  $n$  and  $s = h\hat{e}$ .

On the other hand,  $(2 {}_nF - I)T(2 {}_nF - I) = T$ , since  $T$  and  ${}_nF$  are  $\mathbf{R}$ -linear operators, also  $({}_nF - I)(X) = (I - {}_nF)(X)$  and  $X = {}_nF(X) \oplus ({}_nF - I)(X)$ , hence  ${}_nFT \subset T {}_nF$ . Moreover,  $\lim_n {}_nFx = x$  and

$$\lim_n T {}_nFx = \lim_n {}_nFTx = Tx$$

hence  $\bigcup_n {}_nF(X)$  is a core for  $T$ .

In view of Theorem I.3.9 [28] we get that  $\{E {}_bE|_{E(X)} : b \in \mathbf{R}\}$  is an

$\mathcal{A}_v$  graded resolution of the identity on  $E(X)$ , since  $(h\hat{e})ee_b \leq bee_b$  and  $b(e - ee_b) \leq (h\hat{e})(e - ee_b)$ . Applying Theorem I.3.6 [28] for  ${}_nF$  in place of  $E$  for  $T|_{{}_nF(X)}$  and  $\{{}_nF|_{{}_bE}|_{{}_nF(X)} : b\}$  leads to the formula

$$Tx = \int_{-n}^n d_b E.bx$$

for each  $x \in {}_nF(X)$  and every  $n \in \mathbf{N}$ .

**11. Remark.** Lemma 10 means that

$$\lim_n \int_{-n}^n d_b E.bx = \lim_n \int_{-n}^n d_b E.b|_{{}_nF}x = \lim_n T|_{{}_nF}x = Tx = \int_{-\infty}^{\infty} d_b E.bx$$

for each  $x \in \mathcal{D}(T)$  interpreting the latter integral as improper.

Under conditions imposed in Lemma 10 one says that the function  $h \in \mathcal{Q}(\Lambda)$  represents an affiliated operator  $T\eta\mathbf{A}$ .

Mention that an  $\mathcal{A}_v$  graded projection operator  ${}_bE$  is  $\mathbf{R}$ -homogeneous and  $\mathcal{A}_v$ -additive, hence  $\mathbf{R}$ -linear. Therefore, in the particular case of a real-valued Borel function  $h(b)$  one has  $d_b E.h(b)x = h(b)d_b E.1x$  or simplifying notation  $h(b)d_b Ex$ .

**12. Lemma.** *Suppose that  $\mathbf{A}$  is a quasi-commutative von Neumann algebra over  $\mathcal{A}_v$ , where  $2 \leq v \leq 3$ , so that  $\mathbf{A}$  is isomorphic to  $C(\Lambda, \mathcal{A}_v)$  for some extremely disconnected compact Hausdorff topological space  $\Lambda$ . Then each function  $h \in \mathcal{Q}(\Lambda)$  represents some self-adjoint operator  $T$  affiliated with  $\mathbf{A}$ .*

**Proof.** From §10 it follows that a self-adjoint function  $h$  determines an  $\mathcal{A}_v$  graded resolution of the identity  $\{{}_bE : b \in \mathbf{R}\}$  in  $\mathbf{A}$ . Moreover,  $h\hat{g}_n \in C(\Lambda, \mathcal{A}_v)$ , where  $g_n = e_n - e_{-n}$ ,  $e_n \in C(\Lambda, \mathcal{A}_v)$  represents  ${}_nE$ . Consider an operator  ${}_nT$  corresponding to  $h\hat{g}_n$ . Certainly one has  $(h\hat{g}_m)g_n = h\hat{g}_n$  for each  $n \leq m$ , consequently,  ${}_mT|_{{}_nF} = {}_nT$ , where  ${}_nF$  corresponds to  $g_n$ . Put  $Gx = {}_nTx$  for every vector  $x \in {}_nF(X)$  and  $n \in \mathbf{N}$ . Therefore,  $G$  is an  $\mathbf{R}$  linear  $\mathcal{A}_v$  additive operator on  $\bigcup_{n=1}^{\infty} {}_nF(X) =: \mathcal{K}$ . Therefore, the operator  $G$  is pre-closed and its closure  $T$  is a self-adjoint operator with core  $\mathcal{K}$  as Lemma 6 asserts. For a unitary operator  $U$  in  $\mathbf{A}^*$  and  $x \in {}_nF(X)$  we get  $Ux \in {}_nF(X)$ , hence

$$TUX = {}_nTUX = U|_{{}_nF}Tx.$$

Therefore,  $T\eta\mathbf{A}$  due to Definition 1 and Remark 2.

If  $u \in \mathcal{Q}(\Lambda)$  represents  $T$ , then  $u\hat{g}_n$  represents  $T {}_nF$  in accordance with Lemma 10, consequently,  $h\hat{g}_n = u\hat{g}_n$  for each  $n$ . In view of Lemma 9  $h = u$ , since  $h$  and  $u$  are consistent on a dense subsets in  $\Lambda$ . Thus the function  $h$  represents the self-adjoint operator  $T$ .

**13. Lemma.** *Suppose that  $\{{}_bE : b\}$  is an  $\mathcal{A}_v$  graded resolution of the identity on a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ , also  $\mathbf{A}$  is a quasi-commutative von Neumann algebra over  $\mathcal{A}_v$  so that  $\mathbf{A}$  contains  $\{{}_bE : b\}$ , where  $2 \leq v \leq 3$ . Then there exists a self-adjoint operator  $T$  affiliated with  $\mathbf{A}$  so that*

$$(1) \quad Tx = \int_{-n}^n d_bE.bx$$

for each  $x \in {}_nF(X)$  and every  $n \in \mathbf{N}$ , where  ${}_nF = {}_nE - {}_{-n}E$ , and  $\{{}_bE : b\}$  is the  $\mathcal{A}_v$  graded resolution of the identity for  $T$  given by Lemma 10.

**Proof.** Take a function  $e_b \in C(\Lambda, \mathcal{A}_v)$  corresponding to  ${}_bE$  and a subset  $\Lambda_b = e_b^{-1}(1)$  clopen in  $\Lambda$ . Consider the subsets  $W_+ := \Lambda \setminus \bigcup_b \Lambda_b$  and  $W_- := \bigcap_b \Lambda_b$ . From their definition it follows that  $W_+$  and  $W_-$  are closed in  $\Lambda$ . These two subsets  $W_{\pm}$  are nowhere dense in  $\Lambda$ , since  $\bigwedge_b {}_bE = 0$  and  $\bigvee_b {}_bE = I$ , hence  $W := W_+ \cup W_-$  is nowhere dense in  $\Lambda$ . For a point  $x \in \Lambda \setminus W$  we put  $h(x) := \inf\{b : x \in \Lambda_b\}$ . Given a positive number  $\epsilon > 0$  and  $y \in \Lambda \setminus W$  so that  $h(y) = b$  one gets  $|h(z) - h(y)| \leq \epsilon$  for each  $z \in \Lambda_{b+\epsilon} \setminus \Lambda_{b-\epsilon}$ . This means that the function  $h$  is continuous on  $\Lambda \setminus W$ . Then  $\lim_{y \rightarrow x} h(y) = \pm\infty$  for  $x \in W_{\pm}$ , where  $y \in \Lambda \setminus W$ . Thus the function  $h$  is self-adjoint  $h \in \mathcal{Q}(\Lambda)$  and by Lemma 12 corresponds to a self-adjoint operator  $T$  affiliated with  $\mathbf{A}$ . Certainly we get that  $\{{}_bE : b\}$  is the  $\mathcal{A}_v$  graded resolution of the identity for the operator  $T$  and Formula (1) is valid, since  $\Lambda_b$  is the largest clopen subset in  $\Lambda$  on which the function  $h$  takes values not exceeding  $b$ .

If  $\Psi$  is another such clopen subset and  $e$  is its characteristic function,  $E \in \mathbf{A}$  corresponds to  $e$ , then  $\Psi \subset \Lambda_c$  for each  $c \geq b$ . This leads to the conclusion that  $E \leq \bigwedge_{c>b} {}_cE$  and  $\Psi \subset \Lambda_b$ .

**14. Lemma.** *Suppose that  $T$  is a closed operator on a Hilbert space  $X$  over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  and  $\{{}_bE : b\}$  is an  $\mathcal{A}_v$  graded resolution of the identity on  $X$ , where*

$\mathcal{E} := \bigcup_n {}_nF(X)$  is a core for  $T$  while  ${}_nF = {}_nE - {}_{-n}E$  and

$$(1) \quad Tx = \int_{-n}^n d_b E.bx$$

for each  $x \in {}_nE(X)$  and all  $n$ , then  $T$  is self-adjoint and  ${}_bE$  is the  $\mathcal{A}_v$  graded resolution of the identity for  $T$ .

**Proof.** Formula (1) implies that  $T {}_nF$  is bounded and everywhere defined and is the strong operator limit of finite real-linear combinations of  $\{{}_bE : b\}$ . Thus  ${}_bE(T {}_nF) = (T {}_nF) {}_bE$  and  $T {}_nF$  is self-adjoint, hence  $T$  is self-adjoint by Lemma 6. For each vector  $x \in \mathcal{D}(T)$  there exists a subsequence  $\{n_p : p\}$  of natural numbers and a sequence  $\{p x : p\}$  of vectors such that

$$\lim_p p x = \lim_p {}_{n_p}F p x = x \text{ and } \lim_p T p x = Tx,$$

since  $\mathcal{E}$  is a core for the operator  $T$ . Therefore,  ${}_nFT \subseteq T {}_nF$  for each  $n$ , since

$${}_nFTx = \lim_p {}_nF(T {}_{n_p}F) p x = \lim_p (T {}_{n_p}F) {}_nF p x = \lim_p T {}_nF p x = T {}_nFx.$$

On the other hand, the limits exist:

$$\lim_n (T {}_nF) {}_bEx = \lim_n {}_bE(T {}_nF)x = {}_bETx \text{ and } \lim_n {}_nF {}_bEx = {}_bEx.$$

The operator  $T$  is closed, hence  ${}_bEx \in \mathcal{D}(T)$  and  $T {}_bEx = {}_bETx$ . This leads to the conclusion that  ${}_bET \subseteq T {}_bE$  and  $(2 {}_bE - I)T(2 {}_bE - I) = T$ , since these operators are  $\mathbf{R}$  linear and  $X = {}_bE(X) \oplus ({}_bE - I)(X)$ . Therefore,

$${}_bE(B_{\pm}) = (B_{\pm}){}_bE \text{ for each } b \in \mathbf{R}.$$

Take the quasi-commutative von Neumann algebra

$$\mathbf{G} = \mathbf{G}(T) = cl[alg_{\mathcal{A}_v}\{B_-, B_+; {}_bE : b \in \mathbf{R}\} \oplus alg_{\mathcal{A}_v}\{B_-, B_+; {}_bE : b \in \mathbf{R}\}i].$$

In view of Lemma 10 the operator  $T$  is affiliated with  $\mathbf{G}$ . But Lemma 13 means that there exists a self-adjoint operator  $H$  affiliated with  $\mathbf{G}$  so that

$$Hx = \int_{-n}^n d_b E.bx$$

for each  $x \in {}_nF(X)$  and every natural number  $n$ . Therefore,  $H = T$  and  $\{{}_bE : b\}$  is the  $\mathcal{A}_v$  graded resolution of the identity for  $T$ , since  $H|_{\mathcal{E}} = T|_{\mathcal{E}}$  and  $\mathcal{E}$  is a core for  $T$  and  $H$  simultaneously.

**15. Note.** The quasi-commutative von Neumann algebra  $\mathbf{G} = \mathbf{G}(T)$  generated by  $B_-$  and  $B_+$  in §14 with which the self-adjoint operator  $T$  is affiliated will be called the von Neumann algebra generated by  $T$ .



**16. Theorem.** *Let  $\mathbf{A}$  be a quasi-commutative von Neumann algebra acting on a Hilbert space  $X$  over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ , let also  $\mathcal{Q}(\mathbf{A})$  be a family of all self-adjoint operators affiliated with  $\mathbf{A}$ . Suppose that  $\mathbf{A}$  is isomorphic to  $C(\Lambda, \mathcal{A}_v)$  for an extremely disconnected compact Hausdorff topological space  $\Lambda$  and  $\mathcal{Q}(\Lambda)$  is the family of all self-adjoint functions on  $\Lambda$ . Then*

(a) *there exists a bijective mapping  $\phi$  from  $\mathcal{Q}(\mathbf{A})$  onto  $\mathcal{Q}(\Lambda)$  which is an extension of the isomorphism of  $\mathbf{A}$  with  $C(\Lambda, \mathcal{A}_v)$  for which  $\phi(T) \cdot e$  corresponds to  $TE$  for each projection  $E$  in  $\mathbf{A}$  with  $TE \in \mathbf{A}$ , where  $e \in C(\Lambda, \mathcal{A}_v)$  corresponds to  $E$  and  $((\phi(T) \cdot e)(y) = (\phi(T))(y)$  for  $e(y) = 1$  while  $(\phi(T) \cdot e)(y) = 0$  for  $e(y) = 0$ ;*

(b) *an  $\mathcal{A}_v$  graded resolution  $\{ {}_b E : b \}$  of the identity exists in the quasi-commutative von Neumann subalgebra  $\mathbf{G}$  generated by an operator  $T$  in  $\mathcal{Q}(\mathbf{A})$  so that*

$$(1) \quad Tx = \int_{-n}^n d_b E . bx$$

*for each  $x \in {}_n F(X)$  and every natural number  $n$ , where  ${}_n F = {}_n E - {}_{-n} E$  and  $\bigcup_n {}_n F(X) =: \mathcal{E}$  is a core for  $T$ ;*

(c) *if  $\{ {}_b Q : b \in \mathbf{R} \}$  is an  $\mathcal{A}_v$  graded resolution of the identity on  $X$  so that*

$$(2) \quad Tx = \int_{-n}^n d_b Q . bx$$

*for each  $x \in {}_n P(X)$  and every natural number  $n$ , where  ${}_n P = {}_n Q - {}_{-n} Q$  and  $\bigcup_n {}_n P(X)$  is a core for  $T$ , then  ${}_b E = {}_b Q$  for all  $b$ ;*

(d) *if  $\{ {}_b E : b \in \mathbf{R} \}$  is an  $\mathcal{A}_v$  graded resolution of the identity in  $\mathbf{A}$ , then there exists an operator  $T \in \mathcal{Q}(\mathbf{A})$  for which Formula (1) is valid;*

(e) *if a function  $e_b \in C(\Lambda, \mathcal{A}_v)$  corresponds to  ${}_b E$  and  $\Lambda_b = e_b^{-1}(1)$ , then  $\Lambda_b$  is the largest clopen subset of  $\Lambda$  on which  $\phi(T)$  takes values not exceeding  $b$  in the extended sense.*

The latter theorem is the reformulation of the results obtained above in this section.

**17. Definition.** A closed densely defined operator  $T$  in a Hilbert space

$X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$  will be called normal when two self-adjoint operators  $T^*T$  and  $TT^*$  are equal, where  $2 \leq v$ .

**18. Remark.** For an unbounded  $\mathbf{R}$  homogeneous  $\mathcal{A}_v$  additive operators the following properties are satisfied:

- (1) if  $A \subseteq B$  and  $C \subseteq D$ , then  $A + C \subseteq B + D$ ;
- (2) if  $A \subseteq B$ , then  $CA \subseteq CB$  and  $AC \subseteq BC$ ;
- (3)  $(A + B)C = AC + BC$  and  $CA + CB \subseteq C(A + B)$ .

The latter inclusion does not generally reduce to the equality. For example, an operator  $C$  may be densely defined, but not everywhere, one can take  $A = I$  and  $B = -I$ . This gives  $C(A + B) = 0$  on  $X$ , but  $CA + CB$  is zero only on  $\mathcal{D}(C)$ .

These rules imply that if  $CA \subseteq AC$  for each  $C$  in some family  $\mathcal{Y}$ , then  $TA \subseteq AT$  for each sum of products of operators from  $\mathcal{Y}$ . Apart from algebras of bounded operators a family  $\mathcal{Y}$  may be not extendable to an algebra, since a distributive law generally may be invalid, as it was seen above. Another property is the following:

- (4) if  $\{{}_bT : b \in \Psi\}$  is a net of bounded operators in  $L_q(X)$  so that  $\lim_b {}_bT = T$  in the strong operator topology and  ${}_bTA \subset B {}_bT$  for each  $b$  in a directed set  $\Psi$ , where  $B$  is a closed operator, then  $TA \subseteq BT$ .

Indeed, if  $x \in \mathcal{D}(A)$ , then  ${}_bTx \in \mathcal{D}(B)$  and  $\lim_b B {}_bTx = \lim_b {}_bTAx = TAx$  and hence  $\lim_b {}_bTx = Tx$ . Then  $Tx \in \mathcal{D}(B)$  and  $BTx = TAx$ , since the operator  $B$  is closed, from which property (4) follows. We sum up these properties as the lemma.

**19. Lemma.** *Suppose  $A$  is a closed operator acting in a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$  and  $CA \subseteq AC$  for every operator  $C$  in a self-adjoint subfamily  $\mathcal{Y}$  of  $L_q(X)$ , where  $2 \leq v$ . Then  $TA \subseteq AT$  for each operator  $T$  in the von Neumann algebra over  $\mathcal{A}_v$  generated by  $\mathcal{Y}$ .*

**20. Definition.** When  $A$  is a closed  $\mathbf{R}$  homogeneous  $\mathcal{A}_v$  additive operator in a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ , and  $E$  is an  $\mathcal{A}_v$  graded projection on  $X$  so that  $EA \subseteq AE$  and  $AE$  is a bounded operator on  $X$ , we say that  $E$  is a bounding  $\mathcal{A}_v$  graded projection for  $A$ .

An increasing sequence  $\{{}_nE : n \in \mathbf{N}\}$  such that each  ${}_nE$  is a bounding  $\mathcal{A}_v$  graded projection for  $A$  and  $\bigvee_n {}_nE = I$  will be called a bounding  $\mathcal{A}_v$

graded sequence for  $A$ .

**21. Lemma.** *Let  $E$  be a bounding  $\mathcal{A}_v$  graded projection for a closed densely defined operator  $T$  in a Hilbert space over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ . Then  $E$  is bounding for  $T^*$ ,  $T^*T$  and  $TT^*$  and*

$$(1) (TE)^* = T^*E^*.$$

*Moreover, if  $\{{}_nE : n\}$  is a bounding sequence for  $T$ , then  $\bigcup_n {}_nE(X)$  is a core for  $T$  and  $T^*$  and  $T^*T$  and  $TT^*$ .*

**Proof.** The conditions of this lemma mean that  $ET \subseteq TE$  and  $TE$  is bounded, hence the operator  $ET$  is pre-closed and densely defined and bounded.

Therefore, the operator  $ET$  has the closure  $TE$ , also

(2)  $(TE)^* = (ET)^*$  is closed and hence the operators  $(TE)^*$  and  $(ET)^*$  are closed by Theorem I.3.34 [28]. For each vectors  $x \in E(X)$  and  $y \in \mathcal{D}(T)$  the equality  $\langle Ty; x \rangle = \langle y; (E^*T)^*x \rangle$  is satisfied so that  $x \in \mathcal{D}(T^*)$  and  $T^*x = (ET)^*x$ , consequently,  $T^*E = (E^*T)^*E$ . At the same time we have  $(I - E)\overline{E^*T} = 0$  and hence  $(E^*T)^* = \overline{(E^*T)}^* = (E^*T)^*E = T^*E$ . Therefore,  $E^*T^* \subseteq (TE)^*$  and  $E$  is bounding for  $T^*$  in accordance with Equality (2). Analogously  $E$  is bounding for  $T^*T$  and similarly for  $TT^*$ , since  $E(T^*T) \subseteq (T^*T)E$ .

This implies that  $\{{}_nE : n\}$  is a bounding sequence for  $T^*$ ,  $T^*T$  and  $TT^*$  if it is such for  $T$ . Then  $\lim_n {}_nEx = x$ ,  ${}_nEx \in \mathcal{D}(T)$  and  $\lim_n T {}_nEx = \lim_n {}_nETx = Tx$  for every  $x \in \mathcal{D}(T)$ . Thus  $\bigcup_{n=1}^\infty {}_nE(\mathcal{D}(T))$  is a core for  $T$ , consequently,  $\bigcup_{n=1}^\infty {}_nE(X)$  is a core for  $T$  and  $T^*$ ,  $T^*T$  and  $TT^*$  as well, since  ${}_nE(X) \subseteq \mathcal{D}(T)$  for each natural number  $n$ .

**22. Remark.** Mention that in accordance with Theorem 2.28 [20] an  $\mathcal{A}_v$  graded projection operator can be chosen self-adjoint  $E^* = E$  for a quasi-commutative von Neumann algebra over  $\mathcal{A}_v$ , since  $E$  corresponds to a characteristic function  $e$  which is real-valued.

**23. Theorem.** *Suppose that  $\mathbf{A}$  is a quasi-commutative von Neumann algebra over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ , acting on a Hilbert space  $X$  over  $\mathcal{A}_v$ , also  $T$  and  $B\eta\mathbf{A}$ . Then*

(1) *each finite set of operators affiliated with  $\mathbf{A}$  has a common bounding*

sequence in  $\mathbf{A}$ ;

(2) an operator  $B + T$  is densely defined and pre-closed and its closure is  $B \hat{+} T \eta \mathbf{A}$ ;

(3)  $BT$  is densely defined and pre-closed with the closure  $B \hat{\cdot} T \eta \mathbf{A}$  affiliated with  $\mathbf{A}$ ;

(4)  ${}^j B \hat{\cdot} {}^k T = (-1)^{\kappa(j,k)} {}^k T \hat{\cdot} {}^j B$  for each  $j, k$ , also  $B^* B = B^* \hat{\cdot} B = B B^*$ ;

(5)  $((bI)B \hat{+} T)^* = B^*(b^*I) \hat{+} T^*$  for each quaternion or octonion number  $b \in \mathcal{A}_v$ ;

(6)  $(B \hat{\cdot} T)^* = T^* \hat{\cdot} B^*$ ;

(7) if  $B \subseteq T$ , then  $B = T$ ; if  $B$  is symmetric, then  $B^* = B$ ;

(8) the family  $\mathcal{N}(\mathbf{A})$  of all operators affiliated with  $\mathbf{A}$  forms a quasi-commutative  $*$ -algebra with unit  $I$  under the operations of addition  $\hat{+}$  and multiplication  $\hat{\cdot}$  given by (2, 3).

**Proof.** Take an arbitrary unitary operator  $U$  in the super-commutant  $\mathbf{A}^*$  of  $\mathbf{A}$  (see §II.2.71 [28]). From  $TU = UT$  it follows that  $T^*U = UT^*$ , hence  $T^*\eta\mathbf{A}$ . Then  $(T^*T)U = U(T^*T)$ , since  $U^*U = UU^* = I$ , consequently,  $(T^*T)\eta\mathbf{A}$ . For an  $\mathcal{A}_v$  graded projection  $E$  in  $\mathbf{A}$  an operator  $(2E - I)$  is unitary such that

$$T(2E - I) = (2E - I)T,$$

i.e.  $T(2E - I)x = (2E - I)Tx$  for each  $x \in \mathcal{D}(T) \subset X$ , hence  $ET \subseteq TE$ . In view of Theorem I.3.34 [28] the operator  $T^*T$  is self-adjoint. Take an  $\mathcal{A}_v$  graded resolution  $\{{}_b E : b\}$  of the identity for  $T^*T$  and put  ${}_n F = {}_n E - {}_{-n} E$  for each natural number  $n$ . From Theorem 16 the inclusion  ${}_b E \in \mathbf{A}$  follows. The operator  $T^*T {}_n F$  is bounded and everywhere defined, consequently, the operator  $T {}_n F$  is everywhere defined and closed, since  $T$  is closed and  ${}_n F$  is bounded. In accordance with the closed graph theorem 1.8.6 [12] for  $\mathbf{R}$  linear operators one gets that  $T {}_n F$  is bounded. It can lightly be seen also from the estimate

$\|T {}_n Fx\|^2 = \langle {}_n Fx; T^*T {}_n Fx \rangle \leq \|T^*T {}_n F\| \|x\|^2$ . A sequence  $\{{}_n F : n\}$  of projections is increasing with least upper bound  $I$  for which  ${}_n FT \subseteq T {}_n F$ . Therefore, the limits exist:

$$\lim_n {}_n Fx = x \text{ and } \lim_n T {}_n Fx = \lim_n {}_n FTx = Tx$$

for each vector  $x$  in the domain  $\mathcal{D}(T)$ , consequently,  $\bigcup_{n=1}^{\infty} {}_n F(X)$  is a core

for  $T$  and  ${}_nF$  is a bounding  $\mathcal{A}_v$  graded sequence in  $\mathbf{A}$  for the operator  $T$ .

Let now  $\{{}_nE : n\}$  be a bounding sequence in  $\mathbf{A}$  for  $\{{}_pT : p = 1, \dots, m-1\} \subset \mathbf{A}$  and let  ${}_nF$  be a bounding sequence in  $\mathbf{A}$  for  ${}_mT \in \mathbf{A}$ . Then  $\{{}_nE {}_nF : n\}$  is a bounding sequence in  $\mathbf{A}$  for  ${}_1T, \dots, {}_mT$ , particularly,  $\bigcup_{n=1}^{\infty} {}_nE {}_nF(X)$  is a common core for  ${}_1T, \dots, {}_mT$ . This implies that two operators  $T + B$  and  $T^* + B^*$  are densely defined, but  $T^* + B^* \subseteq (T + B)^*$ , consequently,  $(T + B)^*$  is densely defined and  $T + B$  is pre-closed (see also Theorem I.3.34 [28]).

As soon as  $\{{}_nE : n\}$  is a bounding sequence in  $\mathbf{A}$  for  $T$ ,  $B$ ,  $T^*$  and  $B^*$ , the inclusions are satisfied  ${}_nET \subseteq T {}_nE$  and  ${}_nEB \subseteq B {}_nE$  and hence  ${}_nE(TB) \subseteq (TB) {}_nE$  and  $T {}_nEB \subseteq TB {}_nE$ . The operators  $T {}_nE$  and  $B {}_nE$  are bounded and defined everywhere, consequently,

$$(9) \quad T {}_nEB {}_nE = TB {}_nE$$

(see also Theorem 2.28 [20]). This means that  $\{{}_nE : n\}$  is a bounding sequence for  $TB$  and analogously for  $BT$  and  $B^*T^*$ . That is, the operator  $B^*T^*$  is densely defined. Since  $B^*T^* \subseteq (TB)^*$ , one gets that the operator  $(TB)^*$  is densely defined and  $TB$  is pre-closed. From Formula (9) we infer that

$${}^kT {}^jB {}_nE = (-1)^{\kappa(j,k)} {}^jB {}^kT {}_nE$$

for each  $j, k$ , since  $\mathbf{A}$  is quasi-commutative. Therefore, the operators  $T \hat{\cdot} B$  and  $B \hat{\cdot} T$  agree on their common core  $\bigcup_{n=1}^{\infty} {}_nE(X)$  and inevitably

$$(10) \quad {}^jB \hat{\cdot} {}^kT = (-1)^{\kappa(j,k)} {}^kT \hat{\cdot} {}^jB.$$

The operators  $T^*T$  and  $TT^*$  are self-adjoint, consequently,  $T^*T = T^* \hat{\cdot} T$  and  $TT^* = T \hat{\cdot} T^*$  and  $T^* \hat{\cdot} T = T \hat{\cdot} T^*$ . Then  $U^*x$  and  $Ux \in \mathcal{D}(T + B)$  for each  $x \in \mathcal{D}(T) \cap \mathcal{D}(B) = \mathcal{D}(T + B)$ , consequently,  $U\mathcal{D}(T + B) = \mathcal{D}(T + B)$  and  $(T + B)U = U(T + B)$ . Thus  $(T \hat{\cdot} B)U = U(T \hat{\cdot} B)$ , as well as  $(T \hat{\cdot} B)\eta\mathbf{A}$ .

For each vector  $y \in \mathcal{D}(TB)$  the inclusions follow  $y \in \mathcal{D}(B)$  and  $By \in \mathcal{D}(T)$ . Therefore,  $Uy \in \mathcal{D}(B)$  and  $BUy = UBy \in \mathcal{D}(T)$ , consequently,  $Uy \in \mathcal{D}(TB)$ . Then we get  $U\mathcal{D}(TB) = \mathcal{D}(TB)$ , since  $U^*y \in \mathcal{D}(TB)$ . But  $(TB)Uy = U(TB)y$ , hence  $(T \hat{\cdot} B)Uy = U(T \hat{\cdot} B)y$ , and inevitably  $(T \hat{\cdot} B)\eta\mathbf{A}$ .

Having a bounding  $\mathcal{A}_v$  graded sequence  $\{{}_nE : n\}$  for  $T$  and  $T^*$  one gets  ${}_nET^* \subseteq T^* {}_nE$  and  ${}_nE^*T^* \subseteq (T {}_nE)^*$ , consequently,  $T^* {}_nE$  and  $(T {}_nE)^*$

are bounded everywhere defined extensions of operators  ${}_nET^*$  and  ${}_nE^*T^*$  respectively. In view of Lemma 21 the equality  $(T {}_nE)^* = {}_nE^*T^*$  follows.

We now consider a bounding  $\mathcal{A}_v$  graded sequence  $\{{}_nE : n\}$  for  $T, T^*, B, B^*, ((bI)T\hat{+}B), ((bI)T\hat{+}B)^*, (T\hat{+}B), (T\hat{+}B)^*$  and  $T^*\hat{+}B^*$ . From the preceding demonstration we get the equalities

$$(T^*(b^*I)\hat{+}B^*) {}_nE = (T^*(b^*I)) {}_nE\hat{+}B^* {}_nE = [((bI)T\hat{+}B) {}_nE^*]^* = ((bI)T\hat{+}B)^* {}_nE$$

and

$$(T\hat{+}B)^* {}_nE = [(T\hat{+}B) {}_nE^*]^* = (B^*\hat{+}T^*) {}_nE$$

due to (9, 10). The operators  $((bI)T\hat{+}B)^*$  and  $(T^*(b^*I)\hat{+}B^*)$  agree on their common core  $\bigcup_{n=1}^{\infty} {}_nE(X)$ , consequently,  $((bI)T\hat{+}B)^* = (T^*(b^*I)\hat{+}B^*)$ . Analogously we infer that  $(T\hat{+}B)^* = B^*\hat{+}T^*$ .

Then  $T {}_nE \subseteq B {}_nE$  and hence  $T {}_nE = B {}_nE$  as soon as  $T \subseteq B$  and  $\{{}_nE : n\}$  is a bounding  $\mathcal{A}_v$  graded sequence in  $\mathbf{A}$  for  $T$  and  $B$ . Therefore, the operators  $T$  and  $B$  are consistent on their common core  $\bigcup_{n=1}^{\infty} {}_nE(X)$  and hence  $T = B$ . If  $T$  is symmetric, then  $T \subseteq T^*$  and from the preceding conclusion one obtains  $T = T^*$ .

For any three operators  $T, B, C \in \mathcal{N}(\mathbf{A})$  we take a common bounding  $\mathcal{A}_v$  graded sequence  $\{{}_nE : n\}$  and get

$$(T\hat{+}B)\hat{+}C = T\hat{+}(B\hat{+}C), \text{ since}$$

$$[(T\hat{+}B)\hat{+}C] {}_nE = [T\hat{+}(B\hat{+}C)] {}_nE$$

for all  $n$  (see also [30]). From this Statement (8) of the theorem follows.

**24. Lemma.** *Suppose that  $\{{}_nF : n\}$  is a bounding  $\mathcal{A}_v$  graded sequence for the closed operator  $T$  on a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ , and  $T {}_nF$  is normal for each natural number  $n$ . Then  $T$  is normal.*

**Proof.** In view of Lemma 21 we have the equalities  $(T {}_nF)^* = {}_nF^*T^*$  and  ${}_nFT^* = T^* {}_nF$  and  $T^*T {}_nF = T^* {}_nFT {}_nF$  and  $(T^*T) {}_nF = (TT^*) {}_nF$ . Therefore, the self-adjoint operators  $T^*T$  and  $TT^*$  agree on their common core  $\bigcup_{n=1}^{\infty} {}_nF(X)$ , consequently,  $T^*T = TT^*$ , i.e. the operator  $T$  is normal.

**25. Remark.** The condition  $T^*T = TT^*$  is equivalent to

$$\sum_{j,k} [(T^*)^{i_k} T^{i_j} - T^{i_j} (T^*)^{i_k}] = 0 \text{ on } \mathcal{D}(T^*T) = \mathcal{D}(TT^*), \text{ since } \sum_j \pi^j = I \text{ (see §2).}$$

If  $BT \subseteq TB$  in a Hilbert space  $X$  over the Cayley-Dickson algebra,

then  $\mathbf{B}\mathbf{T} \subseteq \mathbf{T}\mathbf{B}$  for  $\mathbf{B} = \mathbf{i}B = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$  and  $\mathbf{T} = \mathbf{i}T = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$  defined on  $\mathcal{D}(\mathbf{B}) = \mathcal{D}(B) \oplus \mathcal{D}(B)\mathbf{i}$  and  $\mathcal{D}(\mathbf{T}) = \mathcal{D}(T) \oplus \mathcal{D}(T)\mathbf{i}$  with  $\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  (see also §18).

**26. Lemma.** *Let  $BT \subseteq TB$  and  $\mathcal{D}(T) \subseteq \mathcal{D}(B)$ , let also  $T$  be a self-adjoint operator and let  $B$  be a closed operator in a Hilbert space  $X$  over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ . Then  ${}_bEB \subseteq B {}_bE$  for each  ${}_bE$  in the spectral  $\mathcal{A}_v$  graded resolution  $\{{}_bE : b\}$  of  $T$ .*

**Proof.** There is the decomposition  $\text{alg}_{\mathcal{A}_v}(I, B, T, \mathbf{i}I, \mathbf{i}T, \mathbf{i}T) = \text{alg}_{\mathcal{A}_v}(I, B, T) \oplus \text{alg}_{\mathcal{A}_v}(I, B, T)\mathbf{i}$  and this family is defined in the Hilbert space  $X \oplus X\mathbf{i}$ . From Formula 18(3) we have the inclusion  $(BT + B\mathbf{i}I) \subseteq B(T + \mathbf{i}I)$ . Now we consider these operators with domains in  $X \oplus X\mathbf{i}$  denoted by  $\mathcal{D}(T)$ ,  $\mathcal{D}(B)$ , etc. Take an arbitrary vector  $x \in \mathcal{D}(B(T + \mathbf{i}I))$ , then  $x \in \mathcal{D}(T)$  and  $Tx + \mathbf{i}x \in \mathcal{D}(B)$ . But from the suppositions of this lemma we have the inclusion  $\mathcal{D}(T) \subseteq \mathcal{D}(B)$ , consequently,  $Tx \in \mathcal{D}(B)$  and hence  $x \in \mathcal{D}(BT + B(\mathbf{i}I))$  and  $B(T + \mathbf{i}I)x = BTx + B(\mathbf{i}x)$ . Thus  $B(T + \mathbf{i}I) \subseteq BT + B(\mathbf{i}I)$ , consequently,  $B(T + \mathbf{i}I) = BT + B(\mathbf{i}I)$ .

Denote by  $Q_-$  and  $Q_+$  the bounded everywhere defined inverses to  $(T - \mathbf{i}I)$  and  $(T + \mathbf{i}I)$  respectively. In view of 18(1 – 3) and the preceding proof we infer

$Q_{\pm}B = Q_{\pm}B(T \pm \mathbf{i}I)Q_{\pm} = Q_{\pm}(BT \pm B(\mathbf{i}I))Q_{\pm} \subseteq Q_{\pm}(TB \pm B(\mathbf{i}I))Q_{\pm} = Q_{\pm}(T \pm \mathbf{i}I)BQ_{\pm}$  and hence  $(Q_{\pm})B \subseteq B(Q_{\pm})$ . From §10 one has  $Q_+ = (Q_-)^*$  and applying Lemma 19 one gets  $QB \subseteq BQ$  for each element  $Q$  in the von Neumann algebra over  $\mathcal{A}_v$  generated by  $Q_+$  and  $Q_-$ . To an  $\mathcal{A}_v$  graded projection operator  ${}_bE$  on  $X$  an operator  ${}_bE \oplus {}_bE$  on  $X \oplus X\mathbf{i}$  corresponds. Particularly, this means the inclusion  ${}_bEB \subseteq B {}_bE$  in  $X$  for each  $b$ .

**27. Theorem.** *Suppose that  $T$  is an operator on a Hilbert space  $X$  over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ . An operator  $T$  is normal if and only if it is affiliated with a quasi-commutative von Neumann algebra  $\mathbf{A}$  over  $\mathcal{A}_v$ . Moreover, there exists the smallest such algebra  ${}_0\mathbf{A}$ .*

**Proof.** In view of Theorem 23 if an operator is affiliated with a quasi-commutative von Neumann algebra  $\mathbf{A}$  over either the quaternion skew field

or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ , then it is normal. Suppose that an operator  $T$  is normal. Then Lemma 26 is applicable, since  $TT^*T = T^*TT$  and  $\mathcal{D}(T^*T) \subseteq \mathcal{D}(T)$ .

For an  $\mathcal{A}_v$  graded spectral resolution  $\{ {}_bE : b \in \mathbf{R} \}$  this implies that  ${}_bET \subseteq T {}_bE$  for each  $b$ , hence  ${}_nFT \subseteq T {}_nF$  for each natural number  $n$ , where  ${}_nF = {}_nE - {}_{-n}E$ . Then we also have  $T^*T^*T = T^*TT^*$  and  $\mathcal{D}(T^*T) = \mathcal{D}(TT^*) \subseteq \mathcal{D}(T^*)$ , consequently,  ${}_nFT^* \subseteq T^* {}_nF$ . From §23 it follows that the operators  $T {}_nF$  and  ${}_nFT$  are bounded, since such the operator  $T^*T {}_nF = TT^* {}_nF$  is. But then  ${}_nF^*T^* \subseteq (T {}_nF)^*$  and hence both operators  $(T {}_nF)^*$  and  $T^* {}_nF$  are bounded extensions of the densely defined operator  ${}_nFT^*$  when choosing  ${}_bE$  and hence  ${}_nF$  self-adjoint for each  $b$  and  $n$ . Thus one gets the equalities  $(T {}_nF)^* = T^* {}_nF$  and  $(T^* {}_nF)^* = T {}_nF$ . On the other hand, there are the inclusions  $T {}_nFT {}_mF \subseteq TT {}_nF$  and  $T {}_mFT {}_nF \subseteq TT {}_nF$  for each  $n \leq m$ .

The operators  $T {}_nFT {}_mF$  and  $T {}_mFT {}_nF$  are everywhere defined, consequently,  $T {}_nFT {}_mF = TT {}_nF = T {}_mFT {}_nF$ . There are the equalities  $T^* {}_mFT {}_nF = T^*T {}_nF = TT^* {}_nF = T {}_nFT^* {}_mF$ , hence

$${}_0\mathbf{A} := cl[alg_{\mathcal{A}_v}\{ {}_nF, T {}_nF, T^* {}_nF : n \in \mathbf{N} \}]$$

is a quasi-commutative von Neumann algebra over the algebra  $\mathcal{A}_v$ , since there is the inclusion  $alg_{\mathcal{A}_v}\{ {}_nF, T {}_nF, T^* {}_nF : n \in \mathbf{N} \} \subset L_q(X)$ .

Moreover, one has that  $\bigcup_{n=1}^{\infty} {}_nF(X) =: \mathcal{Y}$  is a core for  $T$ , since  $\bigvee_{n=1}^{\infty} {}_nF = I$  and  ${}_nFT \subseteq T {}_nF$ . If  $U$  is a unitary operator in  $({}_0\mathbf{A})^*$  and  $x \in \mathcal{Y}$ , then the equalities  $TUx = TU {}_nFx = T {}_nFUx = UT {}_nFx = UTx$  are fulfilled for some natural number  $n$ . In accordance with Remark 2  $T\eta {}_0\mathbf{A}$  and  $T^*\eta {}_0\mathbf{A}$  also. If  $T\eta \mathcal{R}$ , then  $T^*\eta \mathcal{R}$  and  $T^*T\eta \mathcal{R}$  are affiliations as well. From Note 15 it follows that the self-adjoint operator  $T^*T$  generates a quasi-commutative von Neumann algebra  $\mathbf{A}$  contained in  $\mathcal{R}$ , consequently,  ${}_nF \in \mathcal{R}$  and hence  $T {}_nF, T^* {}_nF \in \mathcal{R}$ . This implies the inclusion  ${}_0\mathbf{A} \subset \mathcal{R}$ .

**28. Definition.** The algebra  ${}_0\mathbf{A}$  from the preceding section will be called the von Neumann algebra over the Cayley-Dickson algebra  $\mathcal{A}_v$  with  $2 \leq v$  generated by the normal operator  $T$ .

**29. Theorem.** Suppose that  $\mathbf{A}$  is a quasi-commutative von Neumann



algebra over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ , also  $\phi$  is an isomorphism of  $\mathbf{A}$  onto  $C(\Lambda, \mathcal{A}_v)$ , where  $\Lambda$  is a compact Hausdorff topological space,  $T\eta\mathbf{A}$ . Then there exists a unique normal function  $\phi(T)$  on  $\Lambda$  so that  $\phi(TE) = \phi(T)\hat{\cdot}\phi(E)$ , when  $E$  is an  $\mathcal{A}_v$  graded bounding projection in  $\mathbf{A}$  for  $T$ , where  $(\phi(T)\hat{\cdot}\phi(E))(z) = \phi(T)(z)\phi(E)(z)$ , if  $\phi(T)(z)$  is defined and zero in the contrary case,  $z \in \Lambda$ . If  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  is the family of all  $\mathcal{A}_v$  valued normal function on  $\Lambda$  and  $f, g \in \mathcal{N}(\Lambda, \mathcal{A}_v)$ , then there are unique normal functions  $\tilde{f}$ ,  $sf$ ,  $fs$ ,  $f\hat{+}g$  and  $f\hat{\cdot}g$  so that  $\tilde{f}(z) = \widetilde{f(z)}$ ,  $(sf)(z) = sf(z)$ ,  $(fs)(z) = f(z)s$ ,  $(f\hat{+}g)(z) = f(z) + g(z)$  and  $(f\hat{\cdot}g)(z) = f(z)\hat{\cdot}g(z)$ , when  $f$  and  $g$  are defined at  $z \in \Lambda$ ,  $s \in \mathcal{A}_v$ . Endowed with the operations  $f \mapsto \tilde{f}$  and  $(s, f) \mapsto sf$  and  $(f, s) \mapsto fs$  and  $(f, g) \mapsto f\hat{+}g$  and  $(f, g) \mapsto f\hat{\cdot}g$ , the family  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  is a quasi-commutative algebra with unit 1 and involution  $f \mapsto \tilde{f}$ , it is associative over the quaternion skew field  $\mathbf{H} = \mathcal{A}_2$  and alternative over the octonion algebra  $\mathbf{O} = \mathcal{A}_3$ . The natural extension of  $\phi$  is a  $*$ -isomorphism of  $\mathcal{N}(\mathbf{A})$  onto  $\mathcal{N}(\Lambda, \mathcal{A}_v)$ .

**Proof.** In view of Theorem 23 an operator  $T$  affiliated with  $\mathbf{A}$  has an  $\mathcal{A}_v$  graded bounding sequence  $\{ {}_nE : n \}$  in  $\mathbf{A}$ . If  $\phi$  has the properties described above, if also  $f$  and  $g$  are normal functions defined at a point  $z$  and corresponding to  $\phi(T)$ , then

$$f(z)\phi({}_nE)(z) = \phi(T {}_nE)(z) = g(z)\phi({}_nE)(z)$$

for every natural number  $n$ . Thus if  $\phi({}_nE)(z) = 1$  for some natural number  $n$ , then certainly  $f(z) = g(z)$ . Applying Lemma 9 we obtain, that  $f = g$ , since the sequence  ${}_nE$  is monotone increasing to  $I$ , while  $f$  and  $g$  agree on a dense subset of  $\Lambda$ . Thus an isomorphism  $\phi$  is unique.

On the other hand, Theorem 16 provides  $\phi(T)$  with the required properties for each  $T \in \mathcal{Q}(\mathbf{A})$ , where  $\mathcal{Q}(\mathbf{A})$  denotes a family of all self-adjoint operators affiliated with  $\mathbf{A}$ . This means that  $\phi$  is defined on  $\mathcal{Q}(\mathbf{A})$ .

For any pair of operators  $T, B$  in  $\mathcal{Q}(\mathbf{A})$  it is possible to choose a bounding sequence  $\{ {}_nE : n \}$  for both  $T$  and  $B$  by Theorem 23. Therefore, the operators  $T {}_nE$ ,  $B {}_nE$ ,  $(T+B) {}_nE$  and  $TB {}_nE$  belong to the quasi-commutative von Neumann algebra  $\mathbf{A}$  over the Cayley-Dickson algebra  $\mathcal{A}_v$  so that

$$(T\hat{+}B) {}_nE = T {}_nE + B {}_nE \text{ and}$$

$$(T\hat{\cdot}B) {}_nE = TB {}_nE. \text{ Therefore, we deduce the equalities}$$

$$\phi(T\hat{+}B)(z)\phi({}_nE)(z) = \phi((T\hat{+}B)_nE)(z) = \phi(T)(z)\phi({}_nE)(z) + \phi(B)(z)\phi({}_nE)(z)$$

and

$$\phi(T\hat{\cdot}B)(z)\phi({}_nE)(z) = \phi((T\hat{\cdot}B)_nE)(z) = \phi(T)(z)\phi(B)(z)\phi({}_nE)(z),$$

when  $\phi(T)$ ,  $\phi(B)$ ,  $\phi(T\hat{+}B)$  and  $\phi(T\hat{\cdot}B)$  are defined at  $z$ . The pairs  $[\phi(T\hat{+}B); \phi(T) + \phi(B)]$  and  $[\phi(T\hat{\cdot}B); \phi(T)\phi(B)]$  agree on dense subsets of  $\Lambda$ . Therefore, images  $\phi(T\hat{+}B)$  and  $\phi(T\hat{\cdot}B)$  are finite, when  $\phi(T)$  and  $\phi(B)$  are defined, hence  $\phi(T\hat{+}B)$  and  $\phi(T\hat{\cdot}B)$  are normal extensions of  $\phi(T) + \phi(B)$  and  $\phi(T)\phi(B)$  correspondingly. Then  $\phi(T)\hat{+}\phi(B)$  and  $\phi(T)\hat{\cdot}\phi(B)$  are defined as normal extensions of  $\phi(T) + \phi(B)$  and  $\phi(T)\phi(B)$  respectively. Each function  $w \in \mathcal{Q}(\Lambda)$  corresponds to some operator  $T \in \mathcal{Q}(\mathbf{A})$  and by Theorem 16 the operations  $\hat{+}$  and  $\hat{\cdot}$  are applicable to all functions in  $\mathcal{Q}(\Lambda)$ . An iterated application of Lemma 9 shows that  $\mathcal{Q}(\Lambda)$  endowed with these operations is a commutative algebra over the real field  $\mathbf{R}$  with unit 1, since each function in  $\mathcal{Q}(\Lambda)$  is real valued. Moreover,  $\phi$  is an isomorphism of  $\mathcal{Q}(\mathbf{A})$  onto  $\mathcal{Q}(\Lambda)$ .

If  $v$  is finite, for each  $j = 0, \dots, 2^v - 1$  the  $\mathbf{R}$ -linear projection operator  $\hat{\pi}^j := \hat{\pi}_v^j : \mathbf{A} \rightarrow \mathbf{A}_j i_j$  is expressible as a sum of products with generators and real constants due to Formulas (1, 2) below so that  $\hat{\pi}^j(A) = i_j A_j = A_j i_j$ :

$$(1) \quad \hat{\pi}^j(A) = (-i_j(A i_j) - (2^v - 2)^{-1} \{-A + \sum_{k=1}^{2^v-1} i_k(A i_k^*)\})/2$$

for each  $j = 1, 2, \dots, 2^v - 1$ ,

$$(2) \quad \hat{\pi}^0(A) = (A + (2^v - 2)^{-1} \{-A + \sum_{k=1}^{2^v-1} i_k(A i_k^*)\})/2,$$

where  $2 \leq v \in \mathbf{N}$ ,

$$(3) \quad A = \sum_j A_j i_j,$$

$A_j \in \mathbf{A}_j$  for each  $j$ ,  $A \in \mathbf{A}$ .

If  $\mathbf{A}$  is embedded into  $L_q(X)$  for a Hilbert space  $X$  over the algebra  $\mathcal{A}_v$ , we can consider projections  $\pi^j : \mathbf{A} \rightarrow \mathbf{A}_j i_j$  and get decomposition (3) so that  $A_j \in \mathbf{A}$  for each  $j$ . The function  $\sum_j \phi(A_j) i_j$  is defined on  $\Lambda \setminus \bigcup_j W_j$  when  $\phi(A_j)$  is defined on  $X \setminus W_j$ . But  $\Lambda \setminus \bigcup_j W_j$  is everywhere dense in  $\Lambda$ , since  $|z| = \sqrt{\sum_j z_j^2}$  is the norm on  $\mathcal{A}_v$ . If  $p \in W_j$ , then

$$\lim_{x \in \Lambda \setminus \bigcup_j W_j, x \rightarrow p} |\phi(A)(x)| = \sqrt{\sum_j \phi(A_j)^2(x)} = \infty.$$

That is  $\sum_j \phi(A_j) i_j$  is an element  $\hat{\sum}_j \phi(A_j) i_j = \phi(A) \in \mathcal{N}(\Lambda, \mathcal{A}_v)$ .

In accordance with Theorem 6.2.26 [6] a topological space  $\Lambda$  is extremely disconnected if and only if for each pair of nonintersecting open subsets  $U$  and  $V$  in  $\Lambda$  their closures always  $cl(U) \cap cl(V) = \emptyset$  do not intersect. By Theorem 8.3.10 [6] if  $(S, \mathcal{U})$  is a uniform space and  $(Y, \mathcal{V})$  is a complete uniform space, then each uniformly continuous mapping  $f : (P, \mathcal{U}_P) \rightarrow (Y, \mathcal{V})$ , where  $P$  is an everywhere dense subset in  $S$  relative to a topology induced by a uniformity  $\mathcal{U}$ , has a uniformly continuous extension  $f : (S, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ .

A set  $\xi^{-1}(0)$  is closed in  $\Lambda$ , when  $\xi \in \mathcal{N}(\Lambda, \mathcal{A}_v)$ . Indeed, a function  $\xi$  is continuous on  $\Lambda \setminus W_\xi$ , consequently,  $\xi^{-1}(0)$  is closed in  $\Lambda \setminus W_\xi$ . At the same time the limit

$$\lim_{x \rightarrow p, x \in \Lambda \setminus W_\xi} |\xi(x)| = \infty$$

is infinite for each  $p \in W_\xi$ , hence  $p \notin cl[\xi^{-1}(0)]$ . This fact implies that the interior  $K := Int[\xi^{-1}(0)]$  is clopen in  $\Lambda$ . Thus the function  $g$  defined such that  $g(x) = 1$  on  $K$ , while  $g(x) := \xi(x)/|\xi(x)|$  on  $\Lambda \setminus (\xi^{-1}(0) \cup W_\xi)$ , is continuous on  $[\Lambda \setminus (\xi^{-1}(0) \cup W_\xi)] \cup K$ . This function  $g$  has a continuous extension  $q \in C(\Lambda, \mathcal{A}_v)$  by Theorems 6.2.26 and 8.3.10 [6], since  $\Lambda \setminus ((\xi^{-1}(0) \setminus K) \cup W_\xi)$  is a dense open subset in  $\Lambda$ .

Indeed, Cantor's cube  $D^m$  is universal for all zero-dimensional spaces of topological weight  $m \geq \aleph_0$ , where  $D$  is a discrete two-element space (see Theorem 6.2.16 [6]). If  $m < \aleph_0$ , then  $\Lambda$  is discrete and this case is trivial. Therefore, if  $m \geq \aleph_0$ , the compact topological space  $\Lambda$  has an embedding into  $D^m$  as a closed subset. Then a uniformity  $\mathcal{U}$  compatible with its topology can be chosen non-archimedean. That is, each pseudo-metric  $\rho$  in a family  $\mathbf{P}$  of all pseudo-metrics inducing a uniformity  $\mathcal{U}$  on  $\Lambda$  satisfies the inequality

$$\rho(x, y) \leq \max(\rho(x, z); \rho(z, y)) \text{ for each } x, y, z \in \Lambda.$$

The function  $g(x)$  has values in the unit sphere in  $\mathcal{A}_v$ , which is closed in the Cayley-Dickson algebra  $\mathcal{A}_v$  and hence is complete. If  $z \in (\xi^{-1}(0) \cup W_\xi) \setminus K$  and  $\{x_\alpha : \alpha \in \Sigma\} \subset [\Lambda \setminus (\xi^{-1}(0) \cup W_\xi)] \cup K$  is a Cauchy net converging to  $z$ , then  $\{x_\alpha : \alpha \in \Sigma\}$  can be chosen such that the limit  $\lim_\alpha g(x_\alpha)$  exists, where  $\Sigma$  is a directed set. For any other Cauchy net  $\{y_\alpha : \alpha \in \Sigma\} \subset [\Lambda \setminus (\xi^{-1}(0) \cup W_\xi)] \cup K$  converging to  $z$  the limit of the net  $\{g(y_\alpha) : \alpha \in \Sigma\}$  exists and will be the same for  $\{g(x_\alpha) : \alpha \in \Sigma\}$ , since each pseudo-metric  $\rho \in \mathbf{P}$  is

non-archimedean and the function  $g$  is continuous on  $[\Lambda \setminus (\xi^{-1}(0) \cup W_\xi)] \cup K$  (see also Propositions 1.6.6 and 1.6.7 and Theorem 8.3.20 [6]).

For a point  $x \in \Lambda \setminus W_\xi$  the equality  $q(x)|\xi(x)| = \xi(x)$  is fulfilled. Since  $|\xi| \in \mathcal{Q}(\Lambda)$ , one gets also normal extensions  $q_j \hat{\cdot} |\xi|$  for  $\xi_j$  defined on  $\Lambda \setminus W_\xi$ . Thus the functions  $\xi, \xi_j$  are in  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  for each  $j$ . Choosing  $A_j \in \mathcal{Q}(\mathbf{A})$  as  $\phi(A_j) = \xi_j$  leads to the equalities  $\phi(\hat{\sum}_j A_j i_j) = \hat{\sum}_j \xi_j i_j = \xi$ . Thus this function  $\phi$  maps  $\mathcal{N}(\mathbf{A})$  onto  $\mathcal{N}(\Lambda, \mathcal{A}_v)$ .

As soon as functions  $f, g \in \mathcal{N}(\Lambda, \mathcal{A}_v)$  are defined on  $\Lambda \setminus W_f$  and  $\Lambda \setminus W_g$  respectively, then components  $f_j$  and  $g_j$  have normal extensions for each  $j$  as follows from the proof above. Therefore, their sum  $f + g$  and product  $fg$  defined on  $\Lambda \setminus (W_f \cup W_g)$  have the normal extensions  $\hat{\sum}_j (f_j \hat{+} g_j) i_j = f \hat{+} g$  and  $\hat{\sum}_{j,k} (f_j \hat{\cdot} g_k) i_j i_k = f \hat{\cdot} g$ .

The set  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  supplied with these operations is the algebra over the Cayley-Dickson algebra  $\mathcal{A}_v$  with  $2 \leq v$ . Moreover, the algebra  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  is quasi-commutative by its construction and with unit 1 and adjoint operation  $f \mapsto \tilde{f}$ . This algebra  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  is associative over the quaternion skew field  $\mathbf{H} = \mathcal{A}_2$  and alternative over the octonion algebra  $\mathbf{O} = \mathcal{A}_3$ , since the multiplication of functions is defined point-wise, while the quaternion skew field  $\mathbf{H}$  is associative and the octonion algebra  $\mathbf{O}$  is alternative. Therefore, the mapping  $\phi$  has an extension up to a  $*$ -isomorphism of  $\mathcal{N}(\mathbf{A})$  onto  $\mathcal{N}(\Lambda, \mathcal{A}_v)$ . For  $T \in \eta \mathbf{A}$  and a bounding  $\mathcal{A}_v$  graded projection  $E$  in  $\mathbf{A}$  for  $T$  we get  $\phi(TE) = \phi(T \hat{\cdot} E) = \phi(T) \hat{\cdot} \phi(E)$ .

**30. Definition.** The spectrum  $sp(T)$  of a closed densely defined operator  $T$  on a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$  with  $2 \leq v$  is the set of all those Cayley-Dickson numbers  $z \in \mathcal{A}_v$  for which an operator  $(T - zI)$  is not a bijective  $\mathbf{R}$  linear  $\mathcal{A}_v$  additive mapping of  $\mathcal{D}(T)$  onto  $X$ , where  $\mathcal{D}(T)$  as usually denotes a  $\mathcal{A}_v$  vector domain of definition of  $T$ .

**31. Remark.** For an operator  $T$  from Definition 30 if  $z \notin sp(T)$ , then  $(T - zI)$  is bijective from  $\mathcal{D}(T)$  onto  $X$  and has an  $\mathbf{R}$  linear  $\mathcal{A}_v$  additive inverse  $B = (T - zI)^{-1} : X \rightarrow \mathcal{D}(T)$ . The graph of  $B$  is closed, since it is such for  $(T - zI)$ . In accordance with the closed graph theorem 1.8.6 [12] this inverse operator  $B$  is bounded. Thus we get  $z \notin sp(T) \Leftrightarrow (T - zI)$  has a bounded inverse from  $X$  onto  $\mathcal{D}(T)$ . If  $T \in \eta \mathbf{A}$  for some quasi-commutative

von Neumann subalgebra in  $L_q(X)$  and  $z \notin sp(T)$ , then  $B \in \mathbf{A}$ . From the boundedness of  $B$  and closedness of  $(T - zI)$  it follows that

$$I = (T - zI)B = (T - zI) \hat{\cdot} B = B \hat{\cdot} (T - zI).$$

Therefore,  $z \notin sp(T)$  is equivalent to:  $(T - zI)$  has an inverse  $B$  in the algebra  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  and  $B \in \mathbf{A}$ .

**32. Proposition.** *Let  $T$  be a normal operator affiliated with a quasi-commutative von Neumann algebra  $\mathbf{A}$  over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ . Then  $sp(T)$  coincides with the range of  $\phi(T)$ , where  $\phi$  denotes the isomorphism of  $\mathcal{N}(\mathbf{A})$  onto  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  extending the isomorphism of  $\mathbf{A}$  with  $C(\Lambda, \mathcal{A}_v)$  (see §29).*

**Proof.** In accordance with Definition 30  $z \notin sp(T)$  if and only if there is a bounded  $B$  inverse to  $(T - zI)$ . For a unitary operator  $U$  in  $\mathbf{A}^*$  one has the equality  $U^*(T - zI)U = (T - zI)$ , since  $(T - zI)\eta\mathbf{A}$ . Therefore,  $U^*BU = B$  for each such unitary operator  $U$  and  $B \in \mathbf{A}$ . But an operator  $B \in \mathbf{A}$  exists so that  $(T - zI) \hat{\cdot} B = I$ . Thus the equality  $\phi(T - zI) \hat{\cdot} \phi(B) = 1$  is satisfied if and only if  $z \notin sp(T)$ . This means that there exists such  $\phi(B) \in C(\Lambda, \mathcal{A}_v)$  if and only if a Cayley-Dickson number  $z$  does not belong to the range of  $\phi(T)$ , hence  $sp(T)$  is the range of  $\phi(T)$ .

**33. Definition.** A self-adjoint operator  $T$  is called positive when

$$\langle Tx; x \rangle \geq 0$$

for each vector  $x \in \mathcal{D}(T)$  in its domain.

**34. Note.** A function  $f - z1$  for  $f \in \mathcal{N}(\Lambda, \mathcal{A}_v)$  and  $z \in \mathcal{A}_v$  has not an inverse in  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  if and only if  $f - z1$  vanishes on some non-void clopen subset in  $\Lambda$ , where  $2 \leq v \leq 3$ . Considering this in  $\mathcal{N}(\mathbf{A})$  we get a non-zero  $\mathcal{A}_v$  graded projection operator  $F \in \mathbf{A}$  so that  $(T - zI) \hat{\cdot} F = 0$ . The latter is equivalent to the existence of a non-zero vector  $x$  on which  $(T - zI)x = 0$ . In this situation one says that  $z$  is in the point spectrum of  $T$ . Thus the spectrum of  $T$  relative to  $\mathcal{N}(\mathbf{A})$  is its point spectrum.

**35. Proposition.** *Let  $T$  be a self-adjoint operator in a Hilbert space  $X$  over either the quaternion skew field or the octonion algebra. Then  $T$  is positive if and only if  $z \geq 0$  for each  $z \in sp(T)$ .*

**Proof.** In view of Lemma 10 it is sufficient to consider the variant, when  $T$  is affiliated with a quasi-commutative von Neumann algebra  $\mathbf{A}$ . There

exists an isomorphism  $\phi$  of  $\mathcal{N}(\mathbf{A})$  onto  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  extending an isomorphism of  $\mathbf{A}$  onto  $C(\Lambda, \mathcal{A}_v)$  (see Theorem 29), where  $\Lambda$  is an extremely disconnected compact Hausdorff topological space. Whenever  $\phi(T)$  is defined on  $\Lambda \setminus W_{\phi(T)}$  and  $\phi(T)(y) < 0$  for some point  $y \in \Lambda \setminus W_{\phi(T)}$ , there exists a non-void clopen subset  $P$  containing  $y$  and a negative constant  $b < 0$  so that  $P \subset \Lambda \setminus W_{\phi(T)}$  and  $\phi(T)(x) < b < 0$  for each  $x \in P$ . If  $F$  is an  $\mathcal{A}_v$  graded projection in  $\mathbf{A}$  corresponding to the characteristic function  $\chi_P$  of  $P$ , then  $F$  is a non-zero  $\mathcal{A}_v$  graded bounding projection for the operator  $T$  so that  $TF \leq bF$ . For each unit vector  $z \in \mathcal{R}(F)$  in the range of  $F$ , the inequality  $\langle Tz; z \rangle \leq b < 0$  would be fulfilled, hence  $T$  would be not positive. This implies the inequality  $sp(T) \geq 0$  if  $T \geq 0$ .

As  $\phi(T)$  has range consisting of non-negative real numbers, its positive square root  $g$  is a normal self-adjoint mapping on  $\Lambda$ . For any  $\phi(B) = g \in \mathcal{Q}(\Lambda)$  the equality  $B^2 = B \hat{\cdot} B = T$  is valid (see Theorem 23). Therefore, for each unit vector  $z \in \mathcal{D}(T)$ , one gets  $z \in \mathcal{D}(B)$  and  $\langle Tz; z \rangle = \langle Bz; Bz \rangle \geq 0$ , hence  $T \geq 0$  whenever  $sp(T) \geq 0$ .

**36. Note.** The set of positive elements in  $\mathcal{N}(\mathbf{A})$  forms a positive cone. Therefore,  $\mathcal{Q}(\mathbf{A})$  is a partially ordered real vector space relative to the partial ordering induced by this cone. But the unit operator  $I$  is not an order unit for  $\mathcal{Q}(\mathbf{A})$  in the considered case.

### 3 Normal operators and homomorphisms

**37. Definition.** A subset  $P$  of a topological space  $W$  is called nowhere dense in  $W$  if its closure has empty interior. A subset  $B$  in  $W$  is called meager or of the first category if it is a countable union  $B = \bigcup_{j=1}^{\infty} P_j$  of nowhere dense subsets  $P_j$  in  $W$ .

There is said that the mapping  $\mathbf{B}(\Phi, \mathbf{R}) \ni f \mapsto f(T) \in L_q(X)$  with the monotone sequential convergence property is  $\sigma$ -normal.

**38. Lemma.** *Suppose that  $\Lambda$  is an extremely disconnected compact Hausdorff topological space. Then each Borel subset of  $\Lambda$  differs from a unique clopen subset by a meager set. Each bounded Borel function  $g$  from  $\Lambda$  into the Cayley-Dickson algebra  $\mathcal{A}_v$  with  $\text{card}(v) \leq \aleph_0$  differs from a unique con-*

tinuous function  $f$  on a meager set. There exists a conjugation-preserving  $\sigma$ -normal homomorphism  $\theta$  from the algebra  $\mathcal{B}(\Lambda, \mathcal{A}_v)$  of bounded Borel functions onto  $C(\Lambda, \mathcal{A}_v)$ . Its kernel  $\ker(\theta)$  consists of all bounded Borel functions vanishing outside a meager set.

**Proof.** As  $\text{card}(v) \leq \aleph_0$  the Cayley-Dickson algebra is separable and of countable topological weight as the topological space relative to its norm topology, since  $\text{card}(\bigcup_{n \in \mathbb{N}} \aleph_0^n) = \aleph_0$ .

Consider the family  $\mathcal{F}$  of all subsets contained in  $\Lambda$  which differ from a clopen subset by a meager set. Take an arbitrary  $V \in \mathcal{F}$  and a clopen subset  $Q$  so that  $(V \setminus Q) \cup (Q \setminus V) =: P$  is meager, i.e.  $\Lambda \setminus V$  and  $\Lambda \setminus Q$  differ by this same meager set. Since  $\Lambda \setminus Q$  is clopen, one gets  $(\Lambda \setminus V) \in \mathcal{F}$ . In addition each open set  $U$  in  $\Lambda$  belongs to  $\mathcal{F}$ , since  $\text{cl}(U)$  is clopen and  $\text{cl}(U) \setminus U$  is nowhere dense in  $\Lambda$ . For a sequence  $\{V_j, Q_j, P_j : j = 0, 1, 2, \dots\}$  of such sets the inclusion is valid:

$$[(\bigcup_{j=1}^{\infty} V_j) \setminus (\bigcup_{j=1}^{\infty} Q_j)] \cup [(\bigcup_{j=1}^{\infty} Q_j) \setminus (\bigcup_{j=1}^{\infty} V_j)] \subseteq \bigcup_{j=1}^{\infty} P_j.$$

The set  $\bigcup_{j=1}^{\infty} P_j$  is meager and the set  $\bigcup_{j=1}^{\infty} Q_j$  is open, hence  $(\bigcup_{j=1}^{\infty} V_j) \in \mathcal{F}$ . Thus the family  $\mathcal{F}$  contains the  $\sigma$ -algebra generated by open subsets, consequently,  $\mathcal{F}$  contains the Borel  $\sigma$ -algebra  $\mathcal{B}(\Lambda)$  of all Borel subsets in  $\Lambda$ .

Theorem 3.9.3 [6] tells that the union  $\bigcup_{j=1}^{\infty} K_j$  of a sequence of nowhere dense subsets  $K_j$  in a Čech-complete topological space  $W$  is a co-dense subset, i.e. its complement set  $W \setminus \bigcup_{j=1}^{\infty} K_j$  is everywhere dense in  $W$ . On the other hand, each topological space metrizable by a complete metric is Čech-complete, also each locally compact Hausdorff space is Čech-complete [6].

This implies that the complement of a meager set is dense in  $\Lambda$ . Therefore, two continuous functions agree on the complement of a meager set only if they are equal. This means that there exists at most one continuous function agreeing with a given bounded Borel function on the complement of a meager set.

Let now  $V$  be a Borel subset of  $\Lambda$ , let also  $g = \chi_V$  be the characteristic function of  $V$ . Take a clopen subset  $Q$  in  $\Lambda$  so that  $(Q \setminus V) \cup (V \setminus Q) =: P$  is a meager subset in  $\Lambda$ . The characteristic function  $f = \chi_Q$  of  $Q$  is continuous and  $(g - f)$  is zero on  $\Lambda \setminus P$ . Therefore, there exists at most one clopen

subset in  $\Lambda$  differing from  $V$  by a meager set. Therefore, a step function being a finite  $\mathcal{A}_v$  vector combination of characteristic functions of disjoint Borel subsets in  $\Lambda$  differs from a continuous function on the complement of a meager set.

The set of step functions is dense in the  $\mathcal{A}_v$  vector space  $\mathcal{B}(\Lambda, \mathcal{A}_v)$  relative to the supremum-norm. This means that if  $g$  is a Borel function  $g : \Lambda \rightarrow \mathcal{A}_v$ ,  $\|g\| := \sup_{x \in \Lambda} |g(x)| < \infty$ , then there exists a sequence of Borel step functions  ${}_ng$  so that  $\lim_{n \rightarrow \infty} \|{}_ng - g\| = 0$ . Let  ${}_nf$  be a sequence of continuous functions agreeing with  ${}_ng$  on the complement of a meager set  $P_n$ . Therefore, the inequality  $\|{}_nf - {}_mf\| \leq \|{}_ng - {}_mg\|$  if fulfilled, since  ${}_nf - {}_mf$  and  ${}_ng - {}_mg$  agree on the complement of a meager set  $P_n \cup P_m$ . The set  $P_n \cup P_m$  is meager and  $|{}_nf(x) - {}_mf(x)| \leq \|{}_ng - {}_mg\|$  for each  $x \in \Lambda \setminus (P_n \cup P_m)$ . Therefore,  $\{{}_nf : n\}$  is a Cauchy sequence converging in supremum norm to a continuous function  $f \in C(\Lambda, \mathcal{A}_v)$ . Therefore,  $f$  and  $g$  agree on  $\Lambda \setminus (\bigcup_{n=1}^{\infty} P_n)$ , where a set  $(\bigcup_{n=1}^{\infty} P_n)$  is meager.

If functions  $g^1$  and  $g^2 \in \mathcal{B}(\Lambda, \mathcal{A}_v)$  differ from  $f^1$  and  $f^2 \in C(\Lambda, \mathcal{A}_v)$  on meager sets  $P^1$  and  $P^2$ , then  $\tilde{g}^1$ ,  $bg^1 + g^2$ ,  $g^1b + g^2$  and  $g^1g^2$  differ from  $\tilde{f}^1$ ,  $bf^1 + f^2$ ,  $f^1b + f^2$  and  $f^1f^2$  on a subset of  $P^1 \cup P^2$ . Thus the mapping  $\theta : \mathcal{B}(\Lambda, \mathcal{A}_v) \ni g \mapsto f \in C(\Lambda, \mathcal{A}_v)$  is a conjugate preserving surjective homomorphism so that  $\theta(g) = 0$  if and only if  $g$  vanishes on the complement of a meager set.

Particularly, if  $\{{}_ng : n\}$  is a monotone increasing sequence in  $\mathcal{B}(\Lambda, \mathbf{R}) \hookrightarrow \mathcal{B}(\Lambda, \mathcal{A}_v)$  of bounded Borel real-valued functions tending point-wise to the bounded Borel function  $g$ , each continuous real-valued function  ${}_nf \in C(\Lambda, \mathcal{A}_v)$  differs from  ${}_ng$  on the meager set  $P_n$ , then  ${}_nf(x) \leq {}_{n+1}f(x)$  for each  $x \in \Lambda \setminus (P_n \cup P_{n+1})$  in a dense set so that  ${}_nf \leq {}_{n+1}f$ . Therefore,  ${}_nf \leq f$  for each natural number  $n$ , while  $f$  differs from  $g$  on the meager set  $P$ . Thus the sequences  $\{{}_ng(x) : n\}$  and  $\{{}_nf(x) : n\}$  tend to  $f(x)$  for each  $x \in \Lambda \setminus (P \cup (\bigcup_{n=1}^{\infty} P_n))$ . This means that the function  $f$  is the least upper bound in  $C(\Lambda, \mathbf{R})$  of the sequence  $\{{}_nf : n\}$  and the homomorphism  $\theta$  is  $\sigma$ -normal.

**39. Corollary.** *Suppose that  $U$  is an open dense subset in an extremely disconnected compact Hausdorff topological space  $\Lambda$  and  $f$  is a continuous*



bounded function  $f : U \rightarrow \mathcal{A}_v$ ,  $\text{card}(v) \leq \aleph_0$ . Then there exists a unique continuous function  $\xi : \Lambda \rightarrow \mathcal{A}_v$  extending  $f$  from  $U$  onto  $\Lambda$ .

**Proof.** Put  $g(x) = f(x)$  for each  $x \in U$ , while  $g(x) = 0$  for each  $x \in \Lambda \setminus U$ , hence  $g$  is a bounded Borel function on  $\Lambda$ . From Lemma 38 we know that a unique continuous function  $\xi : \Lambda \rightarrow \mathcal{A}_v$  exists so that  $\xi$  and  $g$  differ on the complement of a meager set. If  $\xi(x) \neq f(x)$  for some  $x \in U$ , then by continuity of  $f - \xi$  on  $U$  we get that  $f(x) \neq \xi(x)$  on a non-void clopen subset  $W$  of  $U$ . But  $W$  is not meager. This contradicts the choice of  $\xi$ . Thus  $\xi$  is the continuous extension of  $f$  from  $U$  on  $\Lambda$ .

**40. Remark.** If  $\Lambda$  is metrizable, the condition  $\text{card}(v) \leq \aleph_0$  can be dropped in Lemma 38 and Corollary 39 in accordance with §31 in chapter 2 volume 1 [17]. We denote by  $\mathcal{B}_u(\Lambda, \mathcal{A}_v)$  the family of all Borel functions  $f : \Lambda \rightarrow \mathcal{A}_v$ .

**41. Lemma.** Suppose that  $\Lambda$  is an extremely disconnected compact Hausdorff space. Let either  $\Lambda$  be metrizable or  $\text{card}(v) \leq \aleph_0$ . Then a conjugation-preserving surjective homomorphism  $\psi : \mathcal{B}_u(\Lambda, \mathcal{A}_v) \rightarrow \mathcal{N}(\Lambda, \mathcal{A}_v)$  exists so that its kernel  $\ker(\psi)$  consists of all those functions in  $\mathcal{B}_u(\Lambda, \mathcal{A}_v)$  vanishing on the complement of a meager set.

**Proof.** As  $f$  and  $g$  are normal functions defined on  $\Lambda \setminus W_f$  and  $\Lambda \setminus W_g$  respectively and  $f(x) = g(x)$  for each  $x \in \Lambda \setminus (W_f \cup W_g \cup P)$ , where  $P$  is a meager subset of  $\Lambda$ , then  $W_f = W_g$  and  $f = g$  in accordance with Lemma 9, since the set  $(W_f \cup W_g \cup P)$  is meager in  $\Lambda$ , while  $\Lambda \setminus (W_f \cup W_g \cup P)$  is dense in  $\Lambda$ . This implies that at most one normal function can be agreeing with any function on the complement of a meager set.

For each Borel function  $g : \Lambda \rightarrow \mathcal{A}_v$  and each ball  $B(\mathcal{A}_v, y, q) := \{z \in \mathcal{A}_v : |z - y| \leq q\}$  with center  $y$  and radius  $q > 0$ , its inverse image  $g^{-1}(B(\mathcal{A}_v, y, q))$  is a Borel subset  $V_q$  of  $\Lambda$ . In view of Lemma 38 a clopen subset  $Q_n$  exists so that  $(V_n \setminus Q_n) \cup (Q_n \setminus V_n)$  is meager in  $\Lambda$ . Take the Borel function  ${}_n g$  such that  ${}_n g$  is equal to  $g$  on  $V_n$  and is zero on  $\Lambda \setminus V_n$ , consequently, it satisfies the inequality  $\|{}_n g\| \leq n$ . Applying Lemma 38 one gets a continuous function  ${}_n f : \Lambda \rightarrow \mathcal{A}_v$  so that that agrees with  ${}_n g$  on the complement of a meager subset  $P_n$  of  $\Lambda$ .

The function  ${}_n f$  vanishes on the subset  $\Lambda \setminus (V_n \cup P_n)$ , since  ${}_n g$  vanishes

on  $\Lambda \setminus V_n$ . Certainly the set  $\Lambda \setminus (V_n \cup P_n)$  contains the subset  $(\Lambda \setminus Q_n) \setminus (P_n \cup (V_n \setminus Q_n) \cup (Q_n \setminus V_n))$ . The set  $(P_n \cup (V_n \setminus Q_n) \cup (Q_n \setminus V_n))$  is meager and  ${}_nf$  is continuous on  $\Lambda$ , hence this mapping  ${}_nf$  vanishes on  $\Lambda \setminus V_n$ . The inclusion  $V_n \subseteq V_{n+1}$  for each natural number  $n$  implies the inequality  $e_n(x) \leq e_{n+1}(x)$  for each point  $x$  in  $\Lambda$  outside a meager set, where  $e_n := \chi_{Q_n}$  is the characteristic function of  $Q_n$ . But continuity gives  $e_n \leq e_{n+1}$  and hence  $Q_n \subseteq Q_{n+1}$  for every  $n \in \mathbf{N}$ . The functions  ${}_{n+1}g$  and  ${}_ng$  are consistent on  $V_n$ , consequently, continuous functions  ${}_{n+1}f$  and  ${}_nf$  agree on  $Q_n$ . Then the inequality  $n \leq |{}_mf(x)|$  for each  $x \in Q_m \setminus Q_n$  and  $n < m$  is fulfilled, since  $n \leq |{}_mg(y)|$  when  $y \in V_m \setminus V_n$ . The equality  $\bigcup_{n=1}^{\infty} V_n = \Lambda$  leads to the inclusion

$$W_f = \Lambda \setminus (\bigcup_{n=1}^{\infty} Q_n) \subseteq \bigcup_{n=1}^{\infty} (V_n \triangle Q_n),$$

where  $A \triangle B := (A \setminus B) \cup (B \setminus A)$  for two sets. Therefore, the set  $W_f$  is closed and meager, consequently,  $W_f$  is nowhere dense in  $\Lambda$ . If  $f(x) = {}_nf(x)$  for each  $x \in Q_n$  and  $n \in \mathbf{N}$ , then the mapping  $f$  is continuous on  $\Lambda \setminus W_f$ .

For a point  $x \in W_f$  there exists a natural number  $n$  so that  $x \in \Lambda \setminus Q_n$ . When  $y \in \Lambda \setminus (W_f \cup Q_n)$  the inequality  $n \leq |f(y)|$  is valid. Thus the function  $f$  is normal. By the construction above two functions  $f$  and  $g$  agree on the complement of  $W_f \cup (\bigcup_{n=1}^{\infty} P_n)$ , where the latter set is meager. This induces a conjugation-preserving surjective homomorphism  $\psi : \mathcal{B}(\Lambda, \mathcal{A}_v) \ni g \mapsto f \in \mathcal{N}(\Lambda, \mathcal{A}_v)$  with kernel consisting of those Borel functions vanishing on the complement of a meager set in  $\Lambda$ .

**42. Proposition.** *Let  $\mathbf{A}$  be a quasi-commutative von Neumann algebra over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ , let also  ${}_nT$  be a sequence of operators in  $\mathcal{Q}(\mathbf{A})$  with upper bound  ${}_0T$  in  $\mathcal{Q}(\mathbf{A})$ . Then  $\{{}_nT : n = 1, 2, 3, \dots\}$  has a least upper bound  $T$  in  $\mathcal{Q}(\mathbf{A})$ .*

**Proof.** An algebra  $\mathbf{A}$  is  $*$ -isomorphic with  $C(\Lambda, \mathcal{A}_v)$ , where  $\Lambda$  is a compact extremely disconnected Hausdorff space (see Theorem I.2.52 [28]). Take normal functions  ${}_nf$  in  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  representing  ${}_nT$ . Then  $B \hat{+} {}_1T$  is the least upper bound of  $\{{}_nT : n\}$ , when  $B$  is the least upper bound of  $\{{}_nT \hat{+} {}_1T : n\}$ . This means that without loss of generality we can consider  ${}_nT \geq 0$  for each  $n \in \mathbf{N}$ . If a normal function  ${}_nf$  is defined on  $\Lambda \setminus W_n$ , then  $W_n = (W_n)_+$ , hence  $W_n \subseteq W_{n+1}$  for each  $n$ . If  $x \notin W := \bigcup_{n=0}^{\infty} W_n$ , then  ${}_nf(x)$  is de-

defined for each natural number  $n$  and the sequence  $\{{}_nf : n\}$  has an upper bound  ${}_0f(x)$  so that  ${}_0f \leq h$  for each  $h \in \psi^{-1}(\phi({}_0T))$  (see Proposition 32 and Lemma 41). Thus  $\{{}_nf : n\}$  converges to some  $g(x)$ . Put  $g$  to be zero on  $W$ , then  $g$  is a Borel function on  $\Lambda$ . Applying Lemma 41 one gets a normal function  $f \in \mathcal{N}(\Lambda, \mathcal{A}_v)$  agreeing with  $g$  on  $\Lambda \setminus P$ , where  $P$  is a meager subset in  $\Lambda$ . Thus the sequence  $\{{}_nf(x) : n\}$  converges to  $f(x)$  on the dense subset  $\Lambda \setminus (P \cup W)$ . This means that  $f$  is the least upper bound for  $\{{}_nf : n\}$ . Then an element  $T$  in  $\mathcal{Q}(\mathbf{A})$  represented by  $f \in \mathcal{Q}(\Lambda)$  is the least upper bound of  $\{{}_nT : n = 1, 2, \dots\}$ .

**43. Notes.** Using Proposition 42 we can extend our definition of a  $\sigma$ -normal homomorphism to  $\mathcal{N}(\mathbf{A})$ ,  $\mathcal{N}(\Lambda, \mathcal{A}_v)$ ,  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  and  $\mathcal{B}_u(\Lambda, \mathcal{A}_v)$ . In view of Lemma 41 this homomorphism from  $\mathcal{B}_u(\Lambda, \mathcal{A}_v)$  onto  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  is  $\sigma$ -normal.

Consider an increasing sequence  $\{{}_ng : n\}$  of Borel functions on  $\Lambda$  tending point-wise to the Borel function  ${}_0g$  and take the  $\sigma$ -normal functions  ${}_nf$  corresponding to  ${}_ng$ . Then the sequence  $\{{}_nf : n\}$  has the least upper bound  ${}_0f$  in  $\mathcal{N}(\Lambda, \mathcal{A}_v)$ . Indeed, the functions  ${}_ng$  and  ${}_nf$  agree on the complement of a meager set  $P_n$ . Therefore, the limit exists  $\lim_n {}_nf(x) = {}_0f(x)$  for each point  $x \in \Lambda \setminus (\bigcup_{n=0}^{\infty} P_n)$  and the subset  $\Lambda \setminus (\bigcup_{n=0}^{\infty} P_n)$  is dense in  $\Lambda$ . If a function  $h$  is an upper bound for  $\{{}_nf : n = 1, 2, 3, \dots\}$ , then  ${}_nf(x) \leq h(x)$  for each natural number  $n = 1, 2, 3, \dots$  and points  $x \in \Lambda \setminus (\bigcup_{n=0}^{\infty} P_n)$  in the complement of the meager set  $(\bigcup_{n=0}^{\infty} P_n)$ . Thus  ${}_0f(x) \leq h(x)$  for all  $x$  in the complement of this meager set, hence the mapping  $h \hat{+} - {}_0f$  has non-negative values on a dense set. Thus  ${}_0f \leq h$  and  ${}_0f$  is the least upper bound in  $\mathcal{N}(\Lambda, \mathcal{A}_v)$  of the sequence  $\{{}_nf : n = 1, 2, 3, \dots\}$ .

Using Lemma 41 one can define  $g(T)$  for an arbitrary Borel function  $g$  on  $sp(T)$  for any normal operator  $T$  in a Hilbert space either over the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ . In accordance with Theorem 27 operators  $T$ ,  $T^*$  and  $I$  generate a quasi-commutative von Neumann algebra  $\mathbf{A}$  over the algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  such that  $T$  is affiliated with  $\mathbf{A}$ . It is known from Theorems 2.24 and 2.28 [20] that  $\mathbf{A}$  is isomorphic with  $C(\Lambda, \mathcal{A}_v)$  for some extremely disconnected compact Hausdorff topological space  $\Lambda$ . In accordance with Theorem 29 there exists an isomorphism  $\phi$

of  $\mathcal{N}(\mathbf{A})$  onto  $\mathcal{N}(\Lambda, \mathcal{A}_v)$ . As  $\phi(T)$  is defined on  $\Lambda \setminus W$ , the function  $q$  defined to be zero on  $W$  and  $g \circ \phi(T)$  on  $\Lambda \setminus W$  is Borel,  $q \in \mathcal{B}_u(\Lambda, \mathcal{A}_v)$ . But Lemma 41 says that a function  $h$  exists so that  $h \in \mathcal{N}(\Lambda, \mathcal{A}_v)$  and  $h$  agrees with  $q$  on the complement of a meager set in  $\Lambda$ .

We now define  $g(T)$  as  $\phi^{-1}(h)$ .

It may happen that  $sp(T)$  is a subset of a Borel set  $V$  (in  $\mathcal{A}_r$ , for example) and  $g$  is a Borel function on  $V$ , then  $g(T)$  will denote  $g|_{sp(T)}(T)$ , where  $g|_B$  denotes the restriction of  $g$  to a subset  $B \subset V$ .

As usually  $\mathcal{B}(Y, \mathcal{A}_v)$  denotes the algebra of all bounded Borel functions from a topological space  $Y$  into the Cayley-Dickson algebra  $\mathcal{A}_v$ , while  $\mathcal{B}_u(Y, \mathcal{A}_v)$  denotes the algebra of all Borel functions from  $Y$  into  $\mathcal{A}_v$ .

**44. Theorem.** *Let  $\mathbf{A}$  be a quasi-commutative von Neumann algebra over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ ,  $2 \leq v \leq 3$ , so that  $\mathbf{A}$  is generated by a normal operator  $T$  acting on a Hilbert space  $X$  over the algebra  $\mathcal{A}_v$ . Then the mapping  $\theta : g \mapsto g(T)$  of the algebra  $\mathcal{B}_u(sp(T), \mathcal{A}_v)$  into  $\mathcal{N}(\mathbf{A})$  is a  $\sigma$ -normal homomorphism such that  $\theta(1) = I$  and  $\theta(id) = T$ . Moreover, the mapping  $V \mapsto E(V)$  of Borel subsets in  $\mathcal{A}_v$  into  $\mathbf{A}$  is an  $\mathcal{A}_v$  graded projection-valued measure on a Hilbert space  $X$ , where  $E(V) = \chi_V(T)$ ,  $\chi_V$  denotes the characteristic function of  $V \in \mathcal{B}(\mathcal{A}_v)$ . Suppose that  $h : \mathcal{A}_v \rightarrow \mathcal{A}_v$  is a Borel bounded function,  $\mathcal{B}(\mathcal{A}_v, \mathcal{A}_v)$  denotes the algebra over  $\mathcal{A}_v$  of all such Borel bounded functions, then*

$$(1) \quad \|h(T)\| \leq \sup_{y \in \mathcal{A}_v} |h(y)| =: \|h\| \text{ and}$$

$$(2) \quad \langle h(T)x; x \rangle = \int_{\mathcal{A}_v} d\mu_x(y) \cdot h(y)$$

for each vector  $x \in X$  and every Borel bounded function  $h \in \mathcal{B}(\mathcal{A}_v, \mathcal{A}_v)$ , where  $\mu_x(V) \cdot h(y) := \langle E(V) \cdot h(y)x; x \rangle$ . For  $f \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  a vector  $x$  belongs to a domain  $\mathcal{D}(f(T))$  if and only if

$$(3) \quad \int_{\mathcal{A}_v} d\mu_x(y) \cdot |f(y)|^2 =: \|f(T)x\|^2 < \infty$$

and Formula (2) is valid with  $f$  in place of  $h$ .

Let  $\phi$  be an extension on  $\mathcal{N}(\mathbf{A})$  of the isomorphism  $\psi$  of  $\mathbf{A}$  with  $C(\Lambda, \mathcal{A}_v)$ , let also  $\nu_x$  be a regular Borel measure on  $\Lambda$  so that

$$(4) \quad \langle Bx; x \rangle = \int_{\Lambda} d\nu_x(t) \cdot (\phi(B))(t)$$

for each  $B \in \mathbf{A}$ , then  $x \in \mathcal{D}(Q)$  with  $Q \in \mathcal{N}(\mathbf{A})$  if and only if

$$(5) \quad \int_{\Lambda} d\nu_x(t) \cdot |(\phi(Q))(t)|^2 =: \|Qx\|^2 < \infty.$$

If additionally  $T$  is a self-adjoint operator, its spectral resolution is  $\{ {}_bE : b \in \mathbf{R} \}$ , where  ${}_bE = I - E((b, \infty))$ , and  $x \in \mathcal{D}(f(T))$  if and only if

$$(6) \quad \int_{-\infty}^{\infty} d < {}_bE \cdot |f(b)|^2 x; x > < \infty$$

and for such vector  $x \in \mathcal{D}(f(T))$  in a domain of  $f(T)$  the equality

$$(7) \quad < f(T)x; x > = \int_{-\infty}^{\infty} d < {}_bE \cdot f(b)x; x >$$

is valid.

**Proof.** In accordance with Remark 43 there is a  $\sigma$ -normal homomorphism  $\omega$  from  $\mathcal{B}_u(sp(T))$  into  $\mathcal{B}_u(\Lambda, \mathcal{A}_v)$ . On the other hand, the mapping assigning  $h \in \mathcal{N}(\Lambda, \mathcal{A}_v)$  to  $\omega(g)$  is a  $\sigma$ -normal homomorphism, where  $g \in \mathcal{B}_u(sp(T))$ . Therefore, the mapping  $g \mapsto g(T)$  is a  $\sigma$ -normal homomorphism from  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  into  $\mathcal{N}(\mathbf{A})$ . When  $g = 1$  is the constant unit function, the function  $h$  is also constant unit on  $\Lambda$ , consequently,  $g(T) = I$ . When  $g = id$ , then  $h = \phi(T)$ , so that  $id(T) = T$ , where  $id(x) = x$  for each  $x \in \mathcal{A}_v$ .

As  $V$  is a Borel subset of  $\mathcal{A}_v$  and  $g = \chi_V$  is its characteristic function, the identities are satisfied:  $g(T)^* = \tilde{g}(T) = g(T)$  and  $g(T) = g^2(T) = g(T)^2$ . This means that  $g(T)$  is an  $\mathcal{A}_v$  graded projection  $E(V)$  in  $\mathbf{A}$ . Particularly,  $\chi_{\emptyset} = 0$  and  $E(\emptyset) = 0$ ,  $\chi_{\mathcal{A}_v} \equiv 1$ ,  $\chi_{\mathcal{A}_v}(T) = I$  and  $E(\mathcal{A}_v) = I$ .

Recall that  $\mathcal{A}_v$  graded projection valued measures were defined in §§I.2.73 and I.2.58. We use in this section the simplified notation  $E$  instead of  $\hat{\mathbf{E}}$ .

For any countable family  $\{V_j : j\}$  of disjoint Borel subsets in  $\mathcal{A}_v$  and their characteristic functions  ${}_jg := \chi_{V_j}$ , their sums  ${}_ng := {}_1g + \dots + {}_ng$  form an increasing sequence tending point-wise to the characteristic function  $h := \chi_V$ , where  $V = \bigcup_{j=1}^{\infty} V_j$ . Then  $\{\sum_{j=1}^n E(V_j) : n\}$  has the least upper bound  $E(V)$  so that  $E(V) = \sum_{j=1}^{\infty} E(V_j)$ , since  $g \mapsto g(T)$  is a  $\sigma$ -normal homomorphism. Using our notation we get:

$$< h(T)x; x > = < E(V)x; x > = \mu_x(V) = \int_{\mathcal{A}_v} d\mu_x(t) \cdot h(t)$$

hence Equation (2) is valid for Borel step functions  $h$ .

For any bounded Borel function  $h$  on  $\mathcal{A}_v$  with values in  $\mathcal{A}_v$  we have the inequality

$$\|\omega(h)\| \leq \|h\| = \sup_{x \in \mathcal{A}_r} |h(x)|$$

and the function  $f \in \mathcal{N}(\Lambda, \mathcal{A}_v)$  corresponding to  $\omega(h)$  belongs to  $C(\Lambda, \mathcal{A}_v)$  by Lemma 38. We put  $f = \theta(h)$ . Then we infer, that

$$\|h(T)\| = \|\phi^{-1}(f)\| = \|f\| \leq \|h\|,$$

since  $\|f\| \leq \|\omega(h)\| \leq \|h\|$ . Each bounded Borel function in  $\mathcal{B}(\mathcal{A}_v, \mathcal{A}_v)$  is a norm limit of Borel step functions, consequently, Equality (2) is valid for each  $h \in \mathcal{B}(\mathcal{A}_v, \mathcal{A}_v)$ .

We consider now a self-adjoint operator  $T$  and the characteristic function  $g = \chi_{(b, \infty)}$  of  $(b, \infty)$ . Then  $\theta(g)$  is the characteristic function of  $\phi(T)^{-1}((b, \infty))$  which is an open subset contained in  $\Lambda \setminus W$  and hence in  $\Lambda$ . Moreover, the function in  $C(\Lambda, \mathcal{A}_v)$  corresponding to  $\theta(g)$  is the characteristic function  $1 - e_b$  of  $cl[\phi(T)^{-1}((b, \infty))]$ . This means that  $E((b, \infty)) = g(T) \in \mathbf{A}$  corresponds to  $1 - e_b$  and from Theorem 16 one gets  ${}_bE = I - E((b, \infty))$ .

This implies that  $\langle ({}_cE - {}_bE)x; x \rangle = \mu_x((b, c])$  for any  $b \leq c$  and

$$\int_{-\infty}^{\infty} d \langle {}_bE \cdot |f(b)|^2 x; x \rangle = \int d\mu_x(b) \cdot |f(b)|^2$$

for each Borel function  $f \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ , consequently, the last assertions of this theorem reduce to Formulas (2, 3).

Denote by  $q_n := \chi_{V_n}$  the characteristic function of the subset  $V_n := |f|^{-1}([0, n])$ , where  $f \in \mathcal{B}_u(\mathcal{A}_r, \mathcal{A}_v)$  is a Borel function, we put  ${}_nF := q_n(T)$ . Therefore,  ${}_nf(T) = f(T) {}_nF$  due to the first part of the proof, where  ${}_nf = f q_n$ , since the functions  $f$  and  $q_n$  commute,  $f q_n = q_n f$ . This implies that for a vector  $x \in X$  we get the equalities:

$$(6) \quad \|f(T) {}_nF x\|^2 = \langle |f_n|^2(T)x; x \rangle = \int_{V_n} d\mu_x(t) \cdot |f(t)|^2 \text{ and}$$

$$(7) \quad \|f(t) {}_nF x - f(t) {}_mF x\|^2 = \int_{V_n \setminus V_m} d\mu_x(t) \cdot |f(t)|^2.$$

The sequence  $\{{}_nF : n\}$  is increasing with least upper bound  $I$ , since  $\{q_n : n\}$  is the increasing net of non-negative Borel functions tending point-wise to

1. Then we infer that  ${}_nFf(T)x = f(T) {}_nFx$  for each vector  $x$  in a domain  $\mathcal{D}(f(T))$ , since  ${}_nFf(T) \subseteq f(T) {}_nF$ . This implies that the limit exists

$$\lim_n f(T) {}_nFx = f(T)x$$

and Formula (3) follows from (6).

Vise versa, if the integral  $\int_{\mathcal{A}_v} d\mu_x(t) \cdot |f(t)|^2$  converges, then one gets a Cauchy sequence  $\{f(T) {}_nFx : n\}$  due to Formula (7) converging to some vector in the Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ . But  $x \in \mathcal{D}(f(T))$ , since  $\lim_n {}_nFx = x$  and the function  $f(T)$  of the operator  $T$  is closed.

Quite analogous demonstration leads to Formula (5). We have a non-negative measure

$$\mu_x(V) \cdot 1 = \langle E(V) \cdot 1x; x \rangle = \langle E^2(V) \cdot 1x; x \rangle = \langle E(V) \cdot 1x; E^*(V) \cdot 1x \rangle \geq 0$$

for each Borel subset  $V$ . As  $x$  is a vector in the domain  $\mathcal{D}(f(T))$ , the function  $f$  belongs to the Hilbert space  $L^2(\mathcal{A}_v, \mu_x, \mathcal{A}_v)$ , while  $L^2(\mathcal{A}_v, \mu_x, \mathcal{A}_v) \subset L^1(\mathcal{A}_v, \mu_x, \mathcal{A}_v)$ , since  $\mu_x \cdot 1$  is a finite non-negative measure, where  $L^p(\mathcal{A}_v, \mu_x, \mathcal{A}_v)$  with  $1 \leq p < \infty$  denotes an  $\mathcal{A}_v$  vector space which is the norm completion of the family of all step Borel functions  $u$  from  $\mathcal{A}_v$  into  $\mathcal{A}_v$ , where the norm is prescribed by the formula:

$$\|u\| := \sqrt[p]{\int_{\mathcal{A}_v} d\mu_x(t) \cdot |u(t)|^p}.$$

Finally one deduces that

$$\langle f(T)x; x \rangle = \lim_n \langle f(T) {}_nFx; x \rangle = \lim_n \int_{\mathcal{A}_v} q_n(t) d\mu_x(t) \cdot f(t) = \int_{\mathcal{A}_v} d\mu_x(t) \cdot f(t).$$

**45. Remark.** The  $\mathcal{A}_v$  graded projection  $E(V)$  of the preceding theorem will also be referred as the spectral  $\mathcal{A}_v$  graded projection for  $T$  corresponding to the Borel subset  $V$  of  $\mathcal{A}_v$ , where  $2 \leq v \leq 3$ .

**46. Theorem.** Suppose that  $T$  is a normal operator affiliated with a von Neumann algebra acting on a Hilbert space over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  and  $\psi$  is a  $\sigma$ -normal homomorphism of the algebra  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  of Borel functions into  $\mathcal{N}(\mathbf{A})$  so that  $\psi(1) = I$  and  $\psi(id) = T$ , where  $id : \mathcal{A}_v \rightarrow \mathcal{A}_v$  denotes the identity

mapping  $id(t) = t$  on  $\mathcal{A}_v$ . Then  $\psi(f) = f^r(T)$  for each  $f \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ , where  $f^r = f|_{sp(T)}$  denotes the restriction of  $f$  to  $sp(T)$ .

**Proof.** This homomorphism  $\psi$  is adjoint preserving, since  $\psi$  is  $\sigma$ -normal. Positive elements of  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  have positive roots, hence  $\psi$  is order preserving. Moreover,  $\psi : \mathcal{B}(\mathcal{A}_v, \mathcal{A}_v) \rightarrow \mathbf{A}$  and does not increase norm, since  $\psi(1) = I$ .

At first we consider the case when  $T$  is bounded. Put  ${}_0g := \chi_{\mathcal{A}_v \setminus B}$ , where  $B = B(\mathcal{A}_v, 0, 2\|T\|)$  is the closed ball in  $\mathcal{A}_v$  with center 0 and radius  $2\|T\|$ . Then  $0 \leq (2\|T\|)^n {}_0g \leq |id|^n$  for each natural number  $n = 1, 2, 3, \dots$ . Therefore,  $\psi(|id|^n) = T^n$  such that  $0 \leq (2\|T\|)^n \psi({}_0g) \leq |T|^n$ , hence  $\|\psi({}_0g)\| \leq 2^{-n}$  for each natural number  $n$ , consequently,  $\psi({}_0g) = 0$ .

As  ${}_1g$  is the characteristic function of the ball  $B$ , we get  $\psi({}_1g) = I$ , such that  $\psi({}_1gh) = \psi(h)$  for each Borel function  $h \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ .

Consider the restriction  ${}_0h = h|_B$  of  $h$  to  $B$  and put  $\psi^0({}_0h) = \psi(h)$ . Therefore, this mapping  $\psi^0$  is a  $\sigma$ -normal homomorphism of  $\mathcal{B}_u(B, \mathcal{A}_v)$  into  $\mathcal{N}(\mathbf{A})$  with the properties:  $\psi^0(\chi_B) = I$  and  $\psi^0({}_0id) = T$  and  $\psi^0 : \mathcal{B}(B, \mathcal{A}_v) \rightarrow \mathbf{A}$ .

At the same time it is known from the exposition presented above that  $C(B, \mathcal{A}_v)$  is a  $C^*$ -algebra over the algebra  $\mathcal{A}_v$  with unit being the constant function  $\chi_B$ . Therefore, by Theorem 1.3.17 [28] one gets that  $\psi^0(f) = \psi^0(f({}_0id)) = f(T)$  for each continuous function  $f \in C(B, \mathcal{A}_v)$ . It is known from §I.3.17 that the characteristic function  ${}_1h$  of the open subset  $B \setminus sp(T)$  of  $B$  is the point-wise limit of an increasing sequence  $\{{}_nf : n\}$  of positive functions  ${}_nf$  on  $B$ , consequently,  $\psi^0({}_1h)$  is the least upper bound in  $\mathbf{A}$  of the sequence  $\{\psi^0({}_nf) : n\}$ , since the homomorphism  $\psi^0$  is  $\sigma$ -normal. Each function  ${}_nf$  is continuous and vanishes on  $sp(T)$ , consequently,  $\psi^0({}_nf) = {}_nf(T)$  and  ${}_nf(T) = 0$  and hence  $\psi^0({}_1h) = 0$ . For the characteristic function  $\chi_{sp(T)}$  of  $sp(T) \subset B$  one obtains the equalities:  $\psi^0(\chi_{sp(T)}) = I$  and  $\psi^0(\chi_{sp(T)} {}_0h) = \psi^0({}_0h)$  for each  ${}_0h \in \mathcal{B}_u(B, \mathcal{A}_v)$ .

If put  $\psi^1(q|_{sp(T)}) = \psi^0(q)$  for each  $q \in \mathcal{B}_u(B, \mathcal{A}_v)$ , then  $\psi^1 : \mathcal{B}_u(sp(T), \mathcal{A}_v) \rightarrow \mathcal{N}(\mathbf{A})$  is a  $\sigma$ -normal homomorphism so that  $\psi^1(1) = I$  and  $\psi^1(id) = T$  and  $\psi^1(\mathcal{B}(sp(T), \mathcal{A}_v)) \subset \mathbf{A}$ . In view of Theorem I.3.21 [28] applied to the restriction  $\psi^1|_{\mathcal{B}(sp(T), \mathcal{A}_v)}$  the equality  $\psi^1(f) = f(T)$  is valid for each Borel



bounded function  $f \in \mathcal{B}(sp(T), \mathcal{A}_v)$ .

On the other hand, each positive function  $g$  is the point-wise limit of an increasing sequence of positive functions in  $\mathcal{B}(sp(T), \mathbf{R}) \subset \mathcal{B}(sp(T), \mathcal{A}_v)$ . Therefore,  $\psi^1(g) = g(T)$  for each positive Borel function  $g \in \mathcal{B}_u(sp(T), \mathbf{R})$ , since the homomorphism  $\psi^1 : \mathcal{B}_u(sp(T), \mathcal{A}_v) \rightarrow \mathcal{N}(\mathbf{A})$  is  $\sigma$ -normal. Using the decomposition  $h = \sum_j h_j i_j$  of each Borel function  $h \in \mathcal{B}_u(sp(T), \mathcal{A}_v)$  with real-valued Borel functions  $h_j$  and  $h_j = h_j^+ - h_j^-$  with non-negative Borel functions  $h_j^+$  and  $h_j^-$  we infer that  $\psi^1(h) = h(T)$ . This implies that if  $q \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  and  ${}_0q = q|_B$  and  $q^r = q|_{sp(T)}$ , then  $\psi(q) = \psi^0({}_0q) = \psi^1(q^r) = q^r(T)$ .

We take now an arbitrary normal operator  $T \in \mathcal{N}(\mathbf{A})$  and a bounding  $\mathcal{A}_v$  graded projection  $E$  for  $T$  in  $\mathbf{A}$ . The mapping  $\phi$  posing  $(Y \hat{\cdot} E)|_{E(X)} \in \mathcal{N}(\mathbf{A}E)$  acting on  $E(X)$  to  $Y \in \mathcal{N}(\mathbf{A})$  is a  $\sigma$ -normal homomorphism of  $\mathcal{N}(\mathbf{A})$  into  $\mathcal{N}(\mathbf{A}E)$ . Taking the composition of  $\phi$  with  $\psi$  yields a  $\sigma$ -normal homomorphism  $\psi^2 : \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v) \rightarrow \mathcal{N}(\mathbf{A}E)$  mapping 1 onto  $E|_{E(X)}$  and  $id$  onto  $T|_{E(X)}$ . But the composition of  $\phi$  with the mapping  $f \mapsto f^r(T)$  of  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  into  $\mathcal{N}(\mathbf{A}E)$  is another homomorphism. The restriction  $T|_{E(X)}$  is bounded, consequently, from the first part of this proof we infer that

$$(\psi(f) \hat{\cdot} E)|_{E(X)} = \psi^2(T|_{E(X)}) = (f^r(T) \hat{\cdot} E)|_{E(X)}.$$

Theorem 23 states that there exists a common bounding  $\mathcal{A}_v$  graded sequence  $\{{}_nE : n\}$  for  $T$  and  $\psi(f)$  and  $f^r(T)$ , where  $f$  is a given element of  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ . Then we deduce that

$$(\psi(f) \hat{\cdot} {}_nE)|_{{}_nE(X)} = (\psi(f) {}_nE)|_{{}_nE(X)} = (f^r(T) \hat{\cdot} {}_nE)|_{{}_nE(X)} = (f^r(T) {}_nE)|_{{}_nE(X)}$$

so that  $\psi(f) {}_nE = f^r(T) {}_nE$  for each  $n$ . Therefore,  $\psi(f) = f^r(T)$ , since  $\bigcup_{n=1}^{\infty} {}_nE(X)$  is a core for both  $\psi(f)$  and  $f^r(T)$ .

**47. Note.** The procedure of §43 assigning  $g(T) \in \mathbf{A}$  can be applied to  $\mathcal{N}(\Omega, \mathcal{A}_v)$ , where  $\mathbf{Y}$  is another quasi-commutative von Neumann algebra over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  so that  $T$  is affiliated with  $\mathbf{Y}$  and  $\mathbf{Y} \cong C(\Omega, \mathcal{A}_v)$ ,  $\Omega$  is an extremely disconnected compact Hausdorff topological space. The operator in  $\mathcal{N}(\mathbf{A})$  formed in this way is  $g(T) \in \mathcal{N}(\mathbf{Y})$  by Theorem 46.

**48. Corollary.** *Let  $T$  be a normal operator satisfying conditions of Theorem 46 and let  $f$  and  $g$  be in  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ , where  $2 \leq v \leq 3$ . Then*

$$(1) \quad (f \circ g)(T) = f(g(T)).$$

**Proof.** Consider the von Neumann quasi-commutative von Neumann algebra over the algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  generated by  $T$ ,  $T^*$  and  $I$ , then  $g(T) \in \mathcal{N}(\mathbf{A})$  and  $f \mapsto f \circ g$  is a  $\sigma$ -normal homomorphism  $\phi$  of  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  into  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ . Taking the composition of  $\phi$  with the  $\sigma$ -normal homomorphism  $h \mapsto h(T)$  of  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  into  $\mathcal{N}(\mathbf{A})$  leads to a  $\sigma$ -normal homomorphism  $\xi : f \mapsto (f \circ g)(T)$  of  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  into  $\mathcal{N}(\mathbf{A})$  with  $\xi(1) = I$  and  $\xi(id) = g(T)$ . Then Formula (1) follows from Theorem 46.

**49. Proposition.** *Suppose that  $\psi$  is a  $\sigma$ -normal homomorphism of  $\mathcal{N}(\mathbf{A})$  into  $\mathcal{N}(\mathbf{B})$  so that  $\psi(I) = I$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are von Neumann quasi-commutative algebras over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ , where  $2 \leq v \leq 3$ . Then  $\psi(f(T)) = f(\psi(T))$  for each  $T \in \mathcal{N}(\mathbf{A})$  and each  $f \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ .*

**Proof.** Each quasi-commutative algebra  $\mathbf{A}$  over the algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  has the decomposition  $\mathbf{A} = \bigoplus_j \mathbf{A}_j i_j$ , where  $\mathbf{A}_j$  and  $\mathbf{A}_k$  are real isomorphic commutative algebras for each  $j, k = 0, 1, 2, \dots, 2^v - 1$ . On the other hand, the minimal subalgebra  $alg_{\mathbf{R}}(i_0, i_j, i_k)$  for each  $1 \leq j \neq k$  is isomorphic with the quaternion skew field. Therefore, knowing restrictions  $\psi|_{\mathbf{A}_0 \oplus \mathbf{A}_j i_j \oplus \mathbf{A}_k i_k \oplus \mathbf{A}_l i_l}$  for each  $1 \leq j < k$  will induce  $\psi$  on  $\mathbf{A}$ , where  $i_l = i_j i_k$ . Indeed, homomorphisms  $\psi_{j,k} := \psi|_{\mathbf{A}_0 \oplus \mathbf{A}_j i_j \oplus \mathbf{A}_k i_k \oplus \mathbf{A}_l i_l}$  for different  $1 \leq j < k$  and  $1 \leq j' < k'$  are in bijective correspondence:  $\psi_{j',k'} \circ \theta_{j',k'}^{j,k} = \omega_{j',k'}^{j,k} \circ \psi_{j,k}$ , where  $\theta_{j',k'}^{j,k} : \mathbf{A}_0 \oplus \mathbf{A}_j i_j \oplus \mathbf{A}_k i_k \oplus \mathbf{A}_l i_l \rightarrow \mathbf{A}_0 \oplus \mathbf{A}_{j'} i_{j'} \oplus \mathbf{A}_{k'} i_{k'} \oplus \mathbf{A}_{l'} i_{l'}$  and  $\omega_{j',k'}^{j,k} : \mathbf{B}_0 \oplus \mathbf{B}_j i_j \oplus \mathbf{B}_k i_k \oplus \mathbf{B}_l i_l \rightarrow \mathbf{B}_0 \oplus \mathbf{B}_{j'} i_{j'} \oplus \mathbf{B}_{k'} i_{k'} \oplus \mathbf{B}_{l'} i_{l'}$  denote isomorphisms of von Neumann algebras over isomorphic quaternion skew fields.

There exists the mapping  $f \mapsto \psi(f(T))$  of  $\mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  into  $\mathcal{N}(\mathbf{B})$  which is a  $\sigma$ -normal homomorphism so that  $1 \mapsto I$  and  $id \mapsto \psi(T)$ . Applying Theorem 46 we get this assertion.

**50. Corollary.** *Let  $\mathbf{A}$  be a quasi-commutative von Neumann algebra on a Hilbert space  $X$  over either the quaternion skew field or the octonion algebra*

$\mathcal{A}_v$ , where  $2 \leq v \leq 3$ . Let also  $E$  be an  $\mathcal{A}_v$  graded projection in  $\mathbf{A}$  and  $T \eta \mathbf{A}$  and  $f \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$ . Then the identity  $f((T \cdot E)|_{E(X)}) = (f(T) \hat{\cdot} E)|_{E(X)}$  is fulfilled.

**Proof.** Consider the mapping  $B \mapsto (B \hat{\cdot} E)|_{E(X)}$  which is a  $\sigma$ -normal homomorphism  $\psi$  of  $\mathcal{N}(\mathbf{A})$  onto  $\mathcal{N}(\mathbf{A}E|_{E(X)})$  so that  $\psi(I) = E|_{E(X)}$ . But  $E|_{E(X)}$  is the identity operator on  $E(X)$ . Then Proposition 49 implies that  $f((T \cdot E)|_{E(X)}) = f(\psi(T)) = \psi(f(T)) = (f(T) \hat{\cdot} E)|_{E(X)}$ .

**51. Note.** Consider a situation when  $T$  and  $B$  are normal operators, where  $T$  and  $B$  may be unbounded operators whose spectra are contained in the domain of a Borel function  $g \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  and  $g$  has an inverse Borel function  $f$ , where  $2 \leq v \leq 3$ . Then  $g(T) = g(B)$  if and only if  $T = B$ . To demonstrate this mention that if  $g(T) = g(B)$ , then  $T = (f \circ g)(T) = f(g(T)) = f(g(B)) = (f \circ g)(B) = B$  due to Corollary 48.

Observe particularly, that if  $T^2 = B^2$  with positive operators  $T$  and  $B$ , then  $T = B$ . This means that a positive operator has a unique positive square root.

Let  $y$  be a non-zero vector in  $X$  so that  $Ty = by$  for some  $b \in \mathcal{A}_v$  for a normal operator  $T$ , let also  $f \in \mathcal{B}_u(\mathcal{A}_v, \mathcal{A}_v)$  be a Borel function whose domain contains  $sp(T)$ . Suppose in addition that  $T$  is strongly right  $\mathcal{A}_v$  linear, that is by our definition  $T(xu) = (Tx)u$  for each  $x \in X$  and  $u \in \mathcal{A}_v$ . Take an  $\mathcal{A}_v$  graded projection  $E$  with range  $\{x : Tx = bx\}$ , which is closed since  $T$  is a closed operator. Therefore,  $TE = bE$  on  $X$ , since  $ET \subseteq TE$ , and hence

$$f(T)y = [(f(T) \hat{\cdot} E)|_{E(X)}]y = f((T \hat{\cdot} E)|_{E(X)})y = f(bE|_{E(X)})y = f(bI)y$$

in accordance with Corollary 50.

**52. Example.** Let  $X$  be a separable Hilbert space over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  and an orthonormal basis  $\{e_n : n \in \mathbf{N}\}$ , let also  $\mathbf{A}$  be the algebra of bounded diagonal operators  $Te_n = t_n e_n$ , where  $t_n \in \mathcal{A}_v$  for each natural number  $n \in \mathbf{N} = \{1, 2, 3, \dots\}$ . Then  $\mathbf{A}$  is isomorphic with  $C(\Lambda, \mathcal{A}_v)$ , where  $\Lambda = \beta\mathbf{N}$  is the Stone-Ćhech compactification of the discrete space  $\mathbf{N}$  due to Theorems 3.6.1, 6.2.27 and Corollaries 3.6.4 and 6.2.29 [6], since  $\mathbf{N} \subset \Lambda$  and  $\Lambda$  is extremely disconnected. Take points  $s_n$  corresponding to the pure states

$T \mapsto \langle Te_n; e_n \rangle$  of this algebra  $\mathbf{A}$ , where  $n \in \mathbf{N}$ . Then the set  $\{s_n : n \in \mathbf{N}\}$  is dense in  $\Lambda$ , since from  $\langle Te_n; e_n \rangle = 0$  for every  $n$  it follows that  $T = 0$  and each continuous function  $f : \Lambda \rightarrow \mathcal{A}_v$  vanishing on  $\{s_n : n\}$  is zero. Consider the characteristic function  $\chi_{s_n}$  of the singleton  $s_n$ . The projection corresponding to  $\mathcal{A}_v e_n$  lies in  $\mathbf{A}$  and a continuous function corresponds to this projection  $\chi_{s_n} \in C(\Lambda, \mathcal{A}_v)$ . Thus  $s_n$  is an open subset of  $\Lambda$ . Therefore,  $\{s_n : n\}$  is an open dense subset in  $\Lambda$  and its complement  $Z = \Lambda \setminus \{s_n : n\}$  is a closed nowhere dense subset in  $\Lambda$ .

One can define the function  $h(s_n) = t_n \in \mathcal{A}_v$  with  $\lim_n |t_n| = \infty$ , hence this function  $h$  is normal and defined on  $\Lambda \setminus Z$ .

As  $t_n = n\xi_n$  so that  $\xi_n \in \mathcal{A}_r$  with  $|\xi_n| = 1$  for each  $n$  we get a normal function  $f$  corresponding to an operator  $Q$  affiliated with  $\mathbf{A}$  and  $Qe_n = n\xi_n e_n$ . Choosing  $t_n = (n^{1/4} - n)\xi_n$  we obtain a normal function  $g$  corresponding to an operator  $B$  affiliated with  $\mathbf{A}$  so that  $Be_n = (n^{1/4} - n)\xi_n e_n$ . Take the vector  $x = \sum_{n=1}^{\infty} n^{-1} z_n e_n$  with  $z_n \in \{\pm 1, \pm \xi_n, \pm \xi_n^*\}$  for each  $n$ , then  $x \in X$  and  $\nu_x(\{s_n\}) = n^{-2}$ . Thus one gets

$$\int_{\Lambda} d\nu_x(b) \cdot |f(b)|^2 = \infty \text{ and}$$

$$\int_{\Lambda} d\nu_x(b) \cdot |(f+g)(b)|^2 = \sum_{n=1}^{\infty} n^{-3/2} < \infty.$$

The function  $f+g$  is normal when defined on  $\Lambda \setminus Z$ , since  $|(f+g)(s_n)| = n^{1/4} \rightarrow \infty$  and  $f+g$  corresponds to  $Q+B$ . In view of Theorem 44 this vector  $x$  does not belong to  $\mathcal{D}(Q)$ , but  $x \in \mathcal{D}(Q+B)$ . Thus  $Q+B \neq Q+B$ .

Take now  $f$ ,  $Q$  and  $x$  as above and put  $h(s_n) = n^{-3/4}\xi_n$ . The operator  $C$  corresponding to  $h$  is bounded and  $hf$  is normal when defined on  $\Lambda \setminus Z$ . Thus the functions  $hf$  corresponds to the product  $CQ$  and

$$\int_{\Lambda} d\nu_x(b) \cdot |(hf)(b)|^2 = \sum_{n=1}^{\infty} n^{-3/2} < \infty,$$

hence  $x \in \mathcal{D}(CQ)$ . Contrary  $x \notin \mathcal{D}(CQ)$ , since  $x \notin \mathcal{D}(Q)$ . Thus  $CQ \neq CQ$ , consequently, the operator  $CQ$  is not closed. In accordance with Section 23 this product in the reverse order  $QC$  is automatically closed.

**53. Note.** Let  $T$  and  $Q$  be two positive operators affiliated with a quasi-commutative von Neumann algebra  $\mathbf{A}$ , which acts on a Hilbert space  $X$  over

either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$ , where  $2 \leq v \leq 3$ , so that  $\mathbf{A}$  is isomorphic with  $C(\Lambda, \mathcal{A}_v)$ . Positive normal functions  $f$  and  $g$  correspond to these operators  $T$  and  $Q$  such that  $f$  and  $g$  are defined on  $\Lambda \setminus W_f$  and  $\Lambda \setminus W_g$  respectively. Therefore, their sum  $f + g$  is defined on  $\Lambda \setminus (W_f \cup W_g)$  and is normal and corresponds to  $T \hat{+} Q$ , hence

$$0 \leq \int_{\Lambda} d\nu_x(t) \cdot |f(t)|^2 \leq \int_{\Lambda} d\nu_x(t) \cdot |f(t) + g(t)|^2 < \infty$$

for each vector  $x \in \mathcal{D}(T \hat{+} Q)$ . This implies that  $x \in \mathcal{D}(T)$  and symmetrically  $x \in \mathcal{D}(Q)$ , consequently,  $x \in \mathcal{D}(T + Q)$  and  $T + Q = T \hat{+} Q$ .

If drop the condition that  $T$  and  $Q$  are positive, but suppose additionally that  $W_f \cap W_g = \emptyset$ , then two disjoint open subsets  $U_f$  and  $U_g$  in  $\Lambda$  exist containing  $W_f$  and  $W_g$  correspondingly. Therefore,  $cl(U_f) \subset \Lambda \setminus U_g$  and the clopen set  $cl(U_f)$  contains  $W_f$ , since a Hausdorff topological space  $\Lambda$  is extremely disconnected. Thus  $f$  is bounded on  $X \setminus cl(U_f)$  and  $g$  is bounded on  $cl(U_f)$ . Take the  $\mathcal{A}_v$  graded projection in  $\mathbf{A}$  corresponding to the characteristic function  $\chi_{cl(U_f)}$  of  $cl(U_f)$ . Then two operators  $QE$  and  $T(I - E)$  belong to  $L_q(X)$ . But the operator  $TE$  is closed, since  $T$  is closed and  $E$  is bounded, consequently,  $TE = T \hat{+} E$ . We deduce that

$$(T \hat{+} Q)E = TE \hat{+} QE = TE + QE = (T + Q)E \text{ and}$$

$$(T \hat{+} Q)(I - E) = (T + Q)(I - E).$$

Take a vector  $x \in \mathcal{D}(T \hat{+} Q)$ , then  $Ex$  and  $(I - E)x$  are in the domain  $\mathcal{D}(T \hat{+} Q)$ , hence  $Ex$  and  $(I - E)x$  are in  $\mathcal{D}(T + Q)$ , consequently,  $x \in \mathcal{D}(T + Q)$  and inevitably we get that  $T \hat{+} Q = T + Q$ .

**54. Proposition.** *Suppose that  $\mathbf{A}$  is a quasi-commutative von Neumann algebra over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  and  $T\eta\mathbf{A}$ . Let*

$$(1) \quad P(z) = \sum_{k,s, n_1+\dots+n_k \leq n} \{a_{s,n_1} z^{n_1} \dots a_{s,n_k} z^{n_k}\}_{q(2k)}$$

*be a polynomial on  $\mathcal{A}_v$  with  $\mathcal{A}_v$  coefficients  $a_{s,m}$ , where  $k, s \in \mathbf{N}$ ,  $0 \leq n_l \in \mathbf{Z}$  for each  $l$ ,  $n$  is a marked natural number,  $z^0 := 1$ ,  $z \in \mathcal{A}_v$ ,  $q(m)$  is a vector indicating on an order of the multiplication of terms in the curled brackets,*

$a_{1,n_1} \dots a_{1,n_k} \neq 0$  for  $n_1 + \dots + n_k = n$  and constants  $c > 0$  and  $R > 0$  exist so that

$$(2) \quad c|z|^n \leq \left| \sum_{k,s, n_1+\dots+n_k=n} \{a_{s,n_1} z^{n_1} \dots a_{s,n_k} z^{n_k}\}_{q(2k)} \right|$$

for each  $|z| > R$ . Then the operator

$$\sum_{k,s, n_1+\dots+n_k \leq n} \{a_{s,n_1} T^{n_1} \dots a_{s,n_k} T^{n_k}\}_{q(2k)}$$

is closed and equal to  $P(T)$ .

**Proof.** We have that  $\mathbf{A}$  is isomorphic with  $C(\Lambda, \mathcal{A}_v)$ , where  $\Lambda$  is a Hausdorff extremely disconnected compact topological space (see Theorem I.2.52 [28]). An operator  $T$  corresponds to some normal function  $f$  defined on  $\Lambda \setminus W_f$  so that a set  $W_f$  is nowhere dense in  $\Lambda$ . Therefore, the composite function  $P(f(z))$  is defined on  $\Lambda \setminus W_f$  and normal. Thus  $P(f(z))$  corresponds to the polynomial  $P(T)$  of the operator  $T$ . On the other hand, a vector  $x$  is in  $\mathcal{D}(P(T))$  if and only if the integral

$$(3) \quad \int_{\Lambda} d\nu_x(t) \cdot |P(f(t))|^2 < \infty$$

converges. Consider the sets  $\Lambda_m := cl\{t : |f(t)| < m\}$ . Since

$$\lim_{|z| \rightarrow \infty} |P(z)| = \infty$$

and due to Condition (2) there exists a positive number  $m > 0$  such that

$$(4) \quad \frac{1}{2}c|f(t)|^n \leq \frac{1}{2} \left| \sum_{k,s, n_1+\dots+n_k=n} \{a_{s,n_1} f(t)^{n_1} \dots a_{s,n_k} f(t)^{n_k}\}_{q(2k)} \right| \leq |P(f(t))|$$

$\forall t \in \Lambda \setminus (W_f \cup \Lambda_m)$ , consequently,

$$(5) \quad \int_{\Lambda \setminus (W_f \cup \Lambda_m)} d\nu_x(t) \cdot |f(t)|^{2n} < \infty.$$

A measure  $\nu_x$  is non-negative and finite on  $\Lambda$ , the function  $f$  is bounded on  $\Lambda_m$ , consequently,  $f \in L^{2n}(\Lambda, \nu_x, \mathcal{A}_v)$ , hence  $f \in L^k(\Lambda, \nu_x, \mathcal{A}_v)$  for each  $1 \leq k \leq 2n$ . The power  $f^n$  of  $f$  is defined on  $\Lambda \setminus W_f$  and is normal, hence  $f^n$  represents  $\overline{T^n}$ . In view of Theorem 44 the inclusion  $x \in \mathcal{D}(\overline{T^k})$  is fulfilled for each  $k = 1, \dots, n$ , particularly,  $x \in \mathcal{D}(\overline{T}) = \mathcal{D}(T)$ .

We have  $\overline{T} = T$ . Suppose that  $\overline{T^k} = T^k$  for  $k = 1, \dots, j-1$ . As  $y \in \mathcal{D}(\overline{T^j})$ , then  $y \in \mathcal{D}(T)$ . Take a bounding  $\mathcal{A}_v$  graded sequence  ${}_l E$  in  $\mathbf{A}$  for  $\overline{T^{j-1}}$  and

$T$ . Therefore, we deduce that  $\overline{T^j} {}_mE = (\overline{T^{j-1}} \hat{\cdot} T) {}_mE = \overline{T^{j-1}} {}_mET {}_mE$ , consequently,  ${}_mE\overline{T^j}y = \overline{T^j} {}_mEy = \overline{T^{j-1}} {}_mET {}_mEy = \overline{T^{j-1}} {}_mETy$ . On the other hand, the limits exist  $\lim_m {}_mE\overline{T^j}y = \overline{T^j}y$  and  $\lim_m {}_mET^jy = T^jy$ . The operator  $\overline{T^{j-1}}$  is closed, consequently,  $Ty \in \mathcal{D}(\overline{T^{j-1}})$  and  $\overline{T^{j-1}}Ty = \overline{T^j}y$ , hence  $\overline{T^j} \subseteq \overline{T^{j-1}}T$ . By our inductive assumption  $\overline{T^{j-1}} = T^{j-1}$ . Therefore,  $\overline{T^j} \subseteq T^j$  and hence  $\overline{T^j} = T^j$ . Thus by induction we get  $\overline{T^k} = T^k$  for each  $k = 1, \dots, n$ , hence  $x \in \mathcal{D}(T^k)$  for every  $k = 1, \dots, n$ , consequently,  $x \in \mathcal{D}(P(T))$  and inevitably

$$P(T) = \sum_{k,s, n_1+\dots+n_k \leq n} \{a_{s,n_1}T^{n_1} \dots a_{s,n_k}T^{n_k}\}_{q(2k)}.$$

**55. Remark.** For  $v \geq 4$  the Cayley-Dickson algebra  $\mathcal{A}_v$  has divisors of zero. Therefore, we have considered mostly the quaternion and octonion cases  $2 \leq v \leq 3$ . It would be interesting to study further spectral operators over the Cayley-Dickson algebras  $\mathcal{A}_v$  with  $v \geq 4$ , but it is impossible to do this in one article or book if look for comparison on the operator theory over the complex field.

## References

- [1] J.C. Baez. "The octonions". Bull. Amer. Mathem. Soc. **39: 2** (2002), 145-205.
- [2] F. Brackx, R. Delanghe, F. Sommen. "Clifford analysis" (London: Pitman, 1982).
- [3] L.E. Dickson. "The collected mathematical papers". Volumes 1-5 (Chelsea Publishing Co.: New York, 1975).
- [4] N. Dunford, J.C. Schwartz. "Linear operators" (J. Wiley and Sons, Inc.: New York, 1966).
- [5] G. Emch. "Mèchanique quantique quaternionnienne et Relativité restreinte". Helv. Phys. Acta **36** (1963), 739-788.
- [6] R. Engelking. "General topology" (Heldermann: Berlin, 1989).

- [7] J.E. Gilbert, M.A.M. Murray. "Clifford algebras and Dirac operators in harmonic analysis". Cambr. studies in advanced Mathem. **26** (Cambr. Univ. Press: Cambridge, 1991).
- [8] P.R. Girard. "Quaternions, Clifford algebras and relativistic Physics" (Birkhäuser: Basel, 2007).
- [9] K. Gürlebeck, W. Sprössig. "Quaternionic analysis and elliptic boundary value problem" (Birkhäuser: Basel, 1990).
- [10] F. Gürsey, C.-H. Tze. "On the role of division, Jordan and related algebras in particle physics" (World Scientific Publ. Co.: Singapore, 1996).
- [11] M. Junge, Q. Xu. "Representation of certain homogeneous Hilbertian operator spaces and applications". *Invent. Mathematicae* **179: 1** (2010), 75-118.
- [12] R.V. Kadison, J.R. Ringrose. "Fundamentals of the theory of operator algebras" (Acad. Press: New York, 1983).
- [13] I.L. Kantor, A.S. Solodovnikov. "Hypercomplex numbers" (Springer-Verlag: Berlin, 1989).
- [14] R. Killip, B. Simon. "Sum rules and spectral measures of Schrödinger operators with  $L^2$  potentials". *Annals of Mathematics* **170: 2** (2009), 739-782.
- [15] R.S. Krausshar, J. Ryan. "Some conformally flat spin manifolds, Dirac operators and automorphic forms". *J. Math. Anal. Appl.* **325** (2007), 359-376.
- [16] V.V. Kravchenko. "On a new approach for solving Dirac equations with some potentials and Maxwell's system in inhomogeneous media". *Operator Theory* **121** (2001), 278-306.
- [17] K. Kuratowski. "Topology" (Mir: Moscow, 1966).



- [18] S.V. Ludkovsky, F. van Oystaeyen. "Differentiable functions of quaternion variables". Bull. Sci. Math. (Paris). Ser. 2. **127** (2003), 755-796.
- [19] S.V. Ludkovsky. "Differentiable functions of Cayley-Dickson numbers and line integration". J. of Mathem. Sciences **141: 3** (2007), 1231-1298.
- [20] S.V. Ludkovsky. "Algebras of operators in Banach spaces over the quaternion skew field and the octonion algebra". J. Mathem. Sciences **144: 4** (2008), 4301-4366.
- [21] S.V. Ludkovsky. "Residues of functions of octonion variables". Far East Journal of Mathematical Sciences (FJMS), **39: 1** (2010), 65-104.
- [22] S.V. Ludkovsky. "Analysis over Cayley-Dickson numbers and its applications" (LAP Lambert Academic Publishing: Saarbrücken, 2010).
- [23] S.V. Ludkovsky, W. Sproessig. "Ordered representations of normal and super-differential operators in quaternion and octonion Hilbert spaces". Adv. Appl. Clifford Alg. **20: 2** (2010), 321-342.
- [24] S.V. Ludkovsky, W. Sprössig. "Spectral theory of super-differential operators of quaternion and octonion variables", Adv. Appl. Clifford Alg. **21: 1** (2011), 165-191.
- [25] S.V. Ludkovsky, W. Sprössig. "Spectral representations of operators in Hilbert spaces over quaternions and octonions", Complex Variables and Elliptic Equations, online, DOI:10.1080/17476933.2010.538845, 24 pages (2011).
- [26] S.V. Ludkovsky. "Integration of vector hydrodynamical partial differential equations over octonions". Complex Variables and Elliptic Equations, online, DOI:10.1080/17476933.2011.598930, 31 pages (2011).
- [27] S.V. Ludkovsky. "Line integration of Dirac operators over octonions and Cayley-Dickson algebras". Computational Methods and Function Theory, **12: 1** (2012), 279-306.

- [28] S.V. Ludkovsky. "Operator algebras over Cayley-Dickson numbers" (LAP LAMBERT Academic Publishing AG & Co. KG: Saarbrücken, 2011).
- [29] F. van Oystaeyen. "Algebraic geometry for associative algebras". Series "Lect. Notes in Pure and Appl. Mathem." **232** (Marcel Dekker: New York, 2000).
- [30] R.D. Schafer. "An introduction to non-associative algebras" (Academic Press: New York, 1966).
- [31] S. Zelditch. "Inverse spectral problem for analytic domains, II:  $\mathbf{Z}_2$ -symmetric domains". *Advances in Mathematics* **170: 1** (2009), 205-269.